# Domain Range Semigroups and Finite Representations

Jaš Šemrl

RAMiCS 2021

Introduction

Similarity Class with FRP

$R(D, R, *)$ is not Finitely Axiomatisable

Partial Converse and FRP

Problems

# Section 1

## Introduction

- Domain-Range Semigroups to model program correctness [DJS09]
- Functional structures for deterministic programs
- Relational structures for nondeterministic programs

# The Demon and Total Correctness

- Demonic calculus to model the behaviour of programs when the Demon is in charge of nondeterministic choices
- This can be exploited to model total correctness of nondeterministic programs algebraically [HMS20, HŠ21]

# Some Definitions

$$D(S) = \{(x, x) \mid \exists y : (x, y) \in S\}$$
$$R(S) = \{(y, y) \mid \exists x : (x, y) \in S\}$$

$$S; T = \{(x, z) \mid \exists y : (x, y) \in S, (y, z) \in T\}$$
$$S * T = \{(x, y) \in S; T \mid \forall z : (x, z) \in S \Rightarrow (z, z) \in D(T)\}$$

for $S, T \subseteq X \times X$

# Some Definitions

Let $\tau$ be a signature containing predicates, constants, and operations, defined for binary relations.

A $\tau$-structure $\mathcal{S}$ is *proper* if there exists some base $X$, the elements of $\mathcal{S}$ are relations over $X$, and it is closed under the operations in $\tau$.

The *representation class* $R(\tau)$ is the class of all proper $\tau$ structures, closed under isomorphic copies. An isomorphism mapping a $\tau$-structure in $R(\tau)$ to a proper structure is called a *representation*.

# Some Definitions

A representation is *finite* if the image is a proper structure with its elements binary relations over a finite base $X$.

If all finite members of $R(\tau)$ have a finite representation, it is said to have the *Finite Representation Property* (FRP).

$R(\tau)$ is said to be finitely axiomatisable if membership can be axiomatised with finitely many first order formulas.

# Section 2

## Similarity Class with FRP

# Zareckiĭ's Ordered Semigroups

**Theorem [Zar59]**

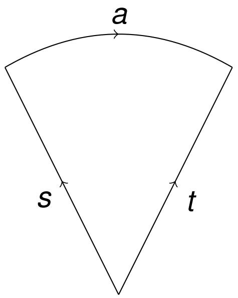
$R(\leq, ;)$ is axiomatised finitely by partial order, associativity, and monotonicity. It's finite members also have the FRP.

Amend $\mathcal{S}$ with an identity element $e$ to obtain $\mathcal{S}'$, for faithfulness.

Then let $\theta : \mathcal{S} \to \wp(\mathcal{S}' \times \mathcal{S}')$ where

$$(s, t) \in a^\theta \Leftrightarrow t \leq s; a$$

for all $s, t \in \mathcal{S}', a \in \mathcal{S}$.



### Theorem [HE13]

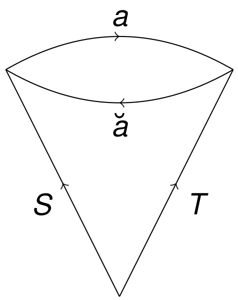
$R(\leq, D, R, \smile, 0, 1, 1', ;)$ is axiomatised by finitely many axioms. It's finite members also have the FRP.

Define the set of good sets $\mathcal{G} \subseteq \wp(\mathcal{S})$ as the set of meet completions.

Then let $\theta : \mathcal{S} \to \wp(\mathcal{G} \times \mathcal{G})$ where

$$(S, T) \in a^\theta \Leftrightarrow (S; a \subseteq T \wedge T; \breve{a} \subseteq S)$$

for all $S, T \in \mathcal{G}, a \in \mathcal{S}$.



# FRP for Wider Similarity Class

1. Proof for FRP for ordered domain algebras [HE13] only requires $\leq, D, R, \smile, ;$ to be in the signature
2. $\leq$ can be defined implicitly as $a \leq b$ if and only if for every representation $\theta$ we have $a^\theta \leq b^\theta$
3. Range can be defined implicitly $R(a) = D(\breve{a})$

## Proposition

FRP holds for the similarity class

$$\{D, \smile, ;\} \subseteq \tau \subseteq \{\leq, \smile, D, R, 0, 1, 1', ;\}$$

# Section 3

## $R(D, R, *)$ is not Finitely Axiomatisable

**Theorem [HM11]**

$R(D, R, ;)$ is not finitely axiomatisable.

1. Define $\leq$ implicitly by saying $a; D(b); c \leq a; c$ and close under transitivity
2. Define unrepresentable structures with $n$-length $\leq$-cycles, for each $n < \omega$
3. Show that the non-principal ultraproduct of these unrepresentable structures is representable

# The Demon Adds an Axiom

Recall, for $S, T \subseteq X \times X$

$$S * T = \{(x, y) \in S; T \mid \forall z : (x, z) \in S \to (z, z) \in D(T)\}$$

This means that for all $S, T \subseteq X \times X$

$$D(S * T) * S = S * D(T)$$

Thus $a * D(b) * c = D(a * b) * a * c$ and the $\leq$ used to show $R(D, R, ;)$ NFA can be finitely axiomatised in $R(D, R, *)$.

# The NFA Proof

**Theorem**

$R(D, R, *)$ cannot be axiomatised by finitely many axioms.

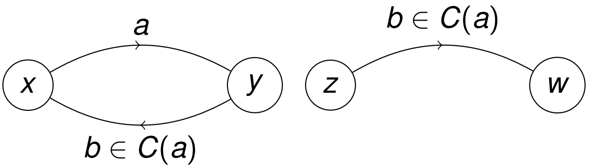
1. Define $\sqsubseteq$ implicitly, using more elaborate axioms
2. Define unrepresentable structures with *n*-length $\sqsubseteq$-cycles, for each $n < \omega$
3. Show that the non-principal ultraproduct of these unrepresentable structures is representable

# Section 4

## Partial Converse and FRP

**Definition**

For every representable $\{D, R, ;\}$- and $\{D, R, *\}$-structure $\mathcal{S}$, define a function $C : \mathcal{S} \to \wp(\mathcal{S})$ such that for all $a, b \in \mathcal{S}$ if $b \in C(a)$, it will be true that in every representation of $\mathcal{S}$, the image of $b$ will be above the converse of the image of $a$.



Why not use this to prove the FRP for converse-free Domain-Range signatures using the Ordered Domain Algebra argument?

Well, that proof relies on

$$(a; b)^{\smile} = \breve{b}; \breve{a}$$

However, it may be the case that

$$C(b); C(a) \subsetneq C(a; b)$$

Section 5

## Problems

# Problems

## Problem

Do converse-free Domain-Range Semigroups have the FRP? How about their demonic cousins?

## Problem

What happens when $+$ is added to the signature? Does the FRP still hold?

## Conjecture

A signature $\tau \subseteq$ *RA* has the FRP if and only if

$$\{-, ;\} \not\subseteq \tau \not\supseteq \{\cdot, ;\}$$

# References


Jules Desharnais, Peter Jipsen, and Georg Struth.
Domain and antidomain semigroups.
In *International Conference on Relational Methods in Computer Science*, pages 73–87. Springer, 2009.

Robin Hirsch and R Egrot.
Meet-completions and representations of ordered domain algebras.
*The Journal of Symbolic Logic*, 2013.

Robin Hirsch and Szabolcs Mikulás.
Axiomatizability of representable domain algebras.
*The Journal of Logic and Algebraic Programming*, 80(2):75–91, 2011.

Robin Hirsch, Szabolcs Mikulás, and Tim Stokes.
The algebra of non-deterministic programs: demonic operators, orders and axioms.
*arXiv preprint arXiv:2009.12081*, 2020.

Robin Hirsch and Jaš Šemrl.
Finite representability of semigroups with demonic refinement.
*Algebra universalis*, 82(2):1–14, 2021.

KA Zareckii.
The representation of ordered semigroups by binary relations, izv. vyss. ucebn. zaved.
*Matematika*, (6):13, 1959.