Computing Least and Greatest Fixed Points in Absorptive Semirings

Matthias Naaf

RWTH AACHEN UNIVERSITY

November 5, RAMiCS 2021

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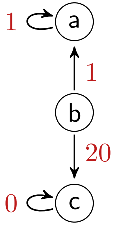
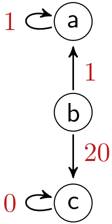
Motivation

Q: Minimal cost of an infinite path?



Motivation

Q: Minimal cost of an infinite path?



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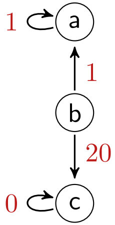

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polynomial equation system

 $= (\mathbb{R}^{\infty}_{\geq 0},\ \min,\ +,\ \infty,\ 0)$

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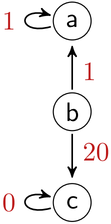

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polynomial equation system

$$X_a = \infty$$

$$X_b = 20$$

$$X_c = 0$$

greatest sol.

🌴 $= (\mathbb{R}_{\geq 0}^{\infty},\ \min,\ +,\ \infty,\ 0)$

Motivation: Semiring Provenance for Logics

Semiring Provenance

- Unify provenance analyses for databases
- Generalize to logics: Semiring semantics for FO, LFP, . . .

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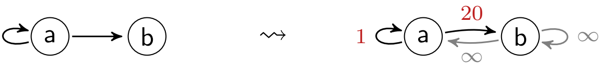
Semiring Semantics

- Idea: Replace Boolean model by semiring annotation:



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Semiring Semantics

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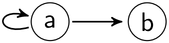
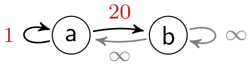


$\rightsquigarrow$



$$G \models Eaa \wedge Eab \quad \rightsquigarrow \quad \pi[\![Eaa \wedge Eab]\!] = 1 + 20$$

Motivation: Semiring Provenance for Logics

Fixed-Point Logic

- $\varphi(v) = [\mathsf{gfp}\, R\, x.\, (\exists y\, Exy \wedge Ry)](v)$
  minimal cost of an infinite path from v (in  *)*

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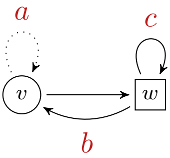
  modify the game so that Player 0 wins (polynomial semiring)



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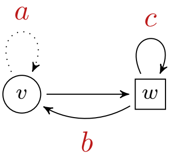

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How to evaluate LFP-formulae?

- least/greatest solutions of PES (in absorptive semirings)

Fixed-Point Iteration?

$$\begin{aligned} X_a &= 1 + X_a \\ X_b &= \min(1 + X_a,\, 20 + X_c) \\ X_c &= 0 + X_c \end{aligned}$$

Fixed-Point Iteration?

$$\mathbf{F}: \boxed{\begin{pmatrix} X_a \\ X_b \\ X_c \end{pmatrix} \mapsto \begin{pmatrix} 1 + X_a \\ \min(1 + X_a,\ 20 + X_c) \\ 0 + X_c \end{pmatrix}}$$

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Faster Computation

Main Result

Let $(K, +, \cdot, 0, 1)$ be an absorptive, fully-continuous semiring.
Given a PES with n variables over K, we can compute:

- $\mathsf{lfp}(\mathbf{F}) = \mathbf{F}^n(0)$.
- $\mathsf{gfp}(\mathbf{F}) = \mathbf{F}^n\big(\,(\mathbf{F}^n(1))^\infty\,\big)$.

We only need a polynomial number of semiring operations

Chapter I

Absorptive Semirings

Semirings with Orders

Commutative Semiring

$(K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ and $(K, \cdot, 1)$ are commutative monoids, $\cdot$ distributes over $+$, $0 \neq 1$ and $0 \cdot a = 0$.

Semirings with Orders

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A semiring is naturally ordered if

$$a \leq b \quad \Longleftrightarrow \quad \exists c.\ a + c = b$$

defines a partial order.

Examples: Boolean semiring, $\mathbb{R}_{\geq 0}$, 🌴, $\mathbb{N}[X]$

Absorptive Semirings

Absorption

A semiring is absorptive if $a + ab = a$ for all a, b.

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Some facts

- Absorptive semirings are idempotent and naturally ordered
- Equivalent definitions:

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- Absorptive semirings are idempotent and naturally ordered
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Remember: Absorption $=$ decreasing multiplication

Absorptive Semirings with Fixed Points

Continuity

An absorptive semiring is K is fully continuous if $\leq$ is a complete lattice satisfying the continuity property:

$$\bigsqcup(a \circ C) = a \circ \bigsqcup C \quad \text{and} \quad \bigsqcap(a \circ C) = a \circ \bigsqcap C$$

for all non-empty chains $C \subseteq K$ and all $a \in K, \circ \in \{+, \cdot\}$.

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for all non-empty chains $C \subseteq K$ and all $a \in K, \circ \in \{+, \cdot\}$.

Infinitary Power

For $a \in K$ we define $a^\infty := \bigsqcap_{n \in \mathbb{N}} a^n$.

Absorptive Semirings with Fixed Points

Examples

- Boolean semiring $(\{0,1\}, \vee, \wedge, 0, 1)$ $\quad a^\infty = a$
- 🌴 $= (\mathbb{R}^{\infty}_{\geq 0}, \min, +, \infty, 0)$ $\quad a^\infty = \begin{cases} 0, & a = 0 \\ \infty, & \text{else} \end{cases}$
- Łukasiewicz semiring $([0,1], \max, \star, 0, 1)$
  with $a \star b = \max(0, a + b - 1)$ $\quad a^\infty = \begin{cases} 1, & a = 1 \\ 0, & \text{else} \end{cases}$
- Any distributive lattice or min-max semiring $\quad a^\infty = a$

Absorptive Semirings with Fixed Points

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Problem: $\mathbb{N}$ and $\mathbb{N}[X]$ not absorptive!

Absorptive Polynomials

Modify $\mathbb{N}[X]$ by

$$2x^2y + xy^2 + 5x^2 + 3z^{10}$$

Absorptive Polynomials

Modify $\mathbb{N}[X]$ by

- dropping coefficients,

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Absorptive Polynomials

Modify $\mathbb{N}[X]$ by

- dropping coefficients,
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$$2x^2y + xy^2 + 5x^2 + 3z^{10}$$

absorbed

Absorptive Polynomials

Modify $\mathbb{N}[X]$ by

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Absorptive Polynomials

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Absorptive polynomials $\mathbb{S}^{\infty}[X]$ are

- always finite (Dickson's lemma),
- the most general absorptive, fully-continuous semiring.

Chapter II

Proof Sketch

Proof Overview: Least Solution

Main Result

Let $(K, +, \cdot, 0, 1)$ be an absorptive, fully-continuous semiring.
Given a PES with n variables over K, we can compute:

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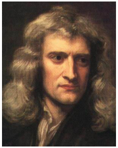
Remark: lfp follows from [Esparza, Kiefer, Luttenberger, ICALP'08]



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Newton's method for $\mathrm{lfp}(\mathbf{F})$ converges in $\mathbf{n}$ steps in idempotent semirings

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Newton's method = fixed-point iteration

Proof Overview: Greatest Solution

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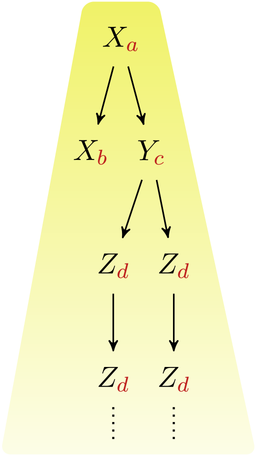
Proof:

1. Express $\mathsf{gfp}(\mathbf{F})$ using derivation trees
2. Apply absorption to derivation trees

Derivation Trees

$$X = aXY + b$$
$$Y = cZ^2$$
$$Z = dZ + e$$



yield: $a \bullet b \bullet c \bullet d^\infty$

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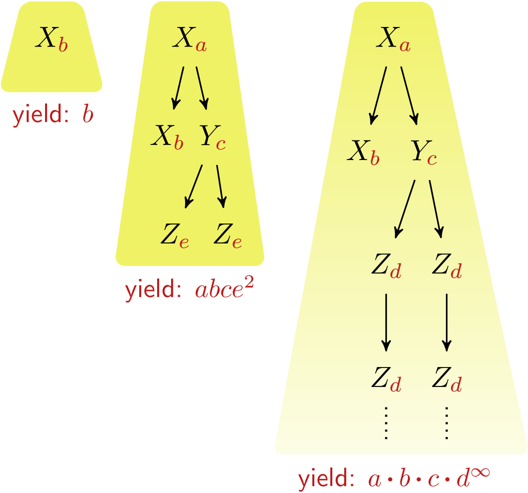

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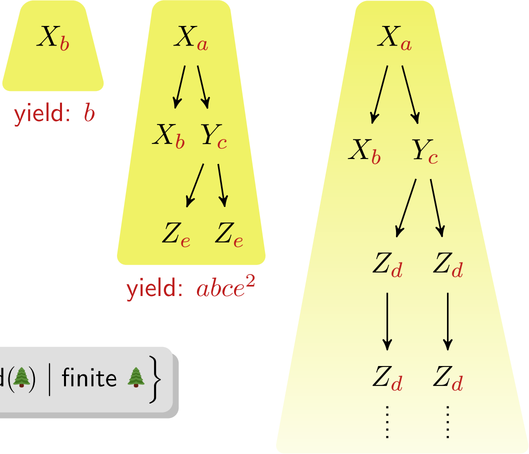

yield: b

yield: $abce^2$

yield: $a \cdot b \cdot c \cdot d^\infty$

lfp $= \sum \left\{ \text{yield}(🌲) \mid \text{finite } 🌲 \right\}$

Derivation Trees

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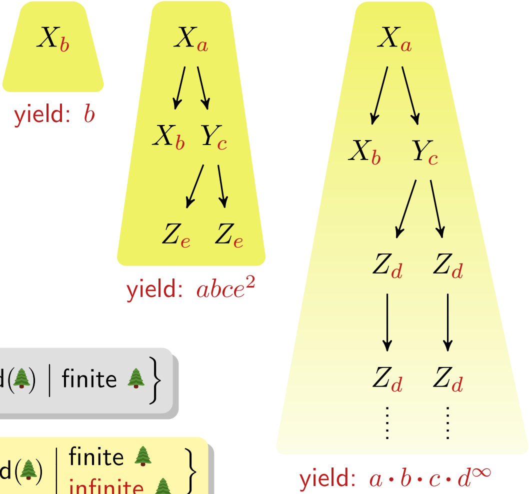
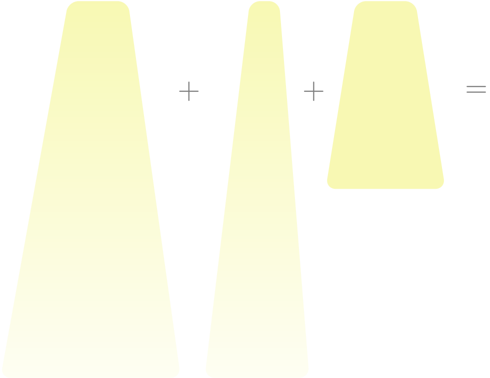


lfp $= \sum \{$ yield(🌲) | finite 🌲 $\}$

gfp $= \sum \{$ yield(🌲) | finite 🌲, infinite 🌲 $\}$

Cutting Down Derivation Trees

Observation: Prefixes of 🌲 correspond to iteration steps.



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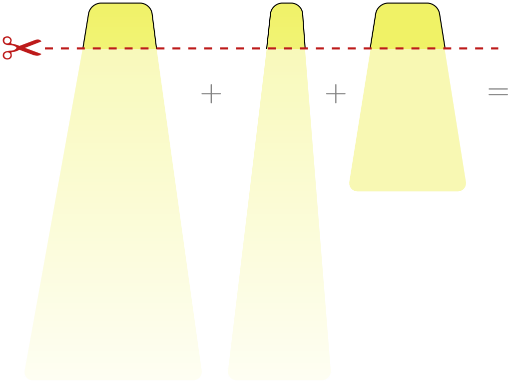

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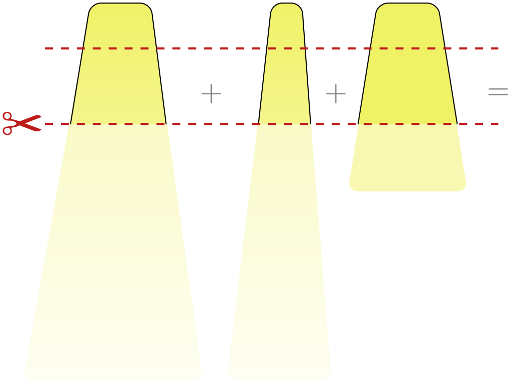

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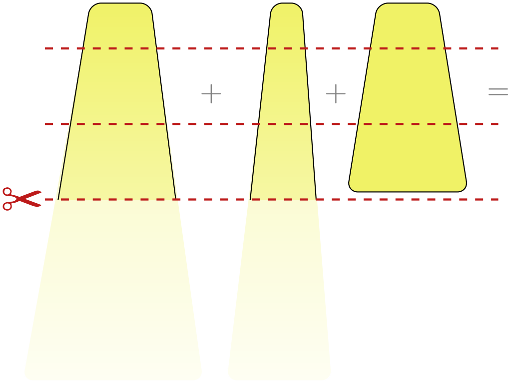

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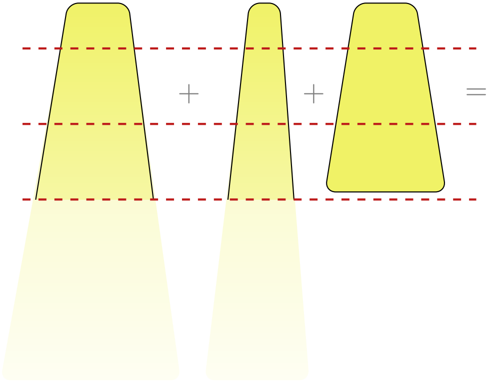

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$$\bigsqcap_{n \in \mathbb{N}} : \quad \sum \Big\{ \text{yield}(🌲) \;\Big|\; \text{finite/infinite } 🌲 \Big\} \quad = \quad \text{gfp}(\mathbf{F}) \qquad \blacksquare$$

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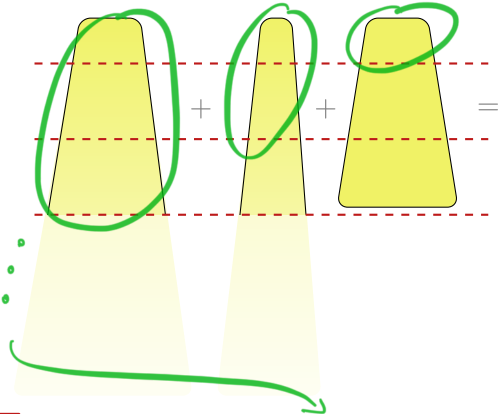

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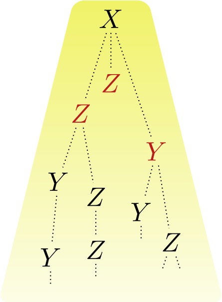

complicated tree 🌳

?

nice tree 🌲

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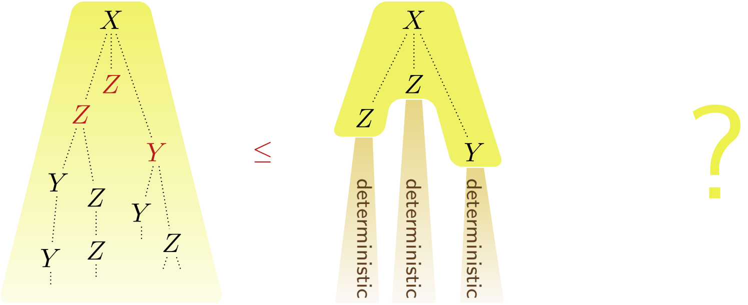

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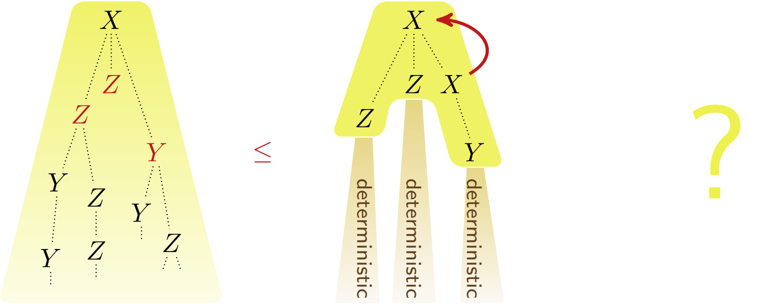

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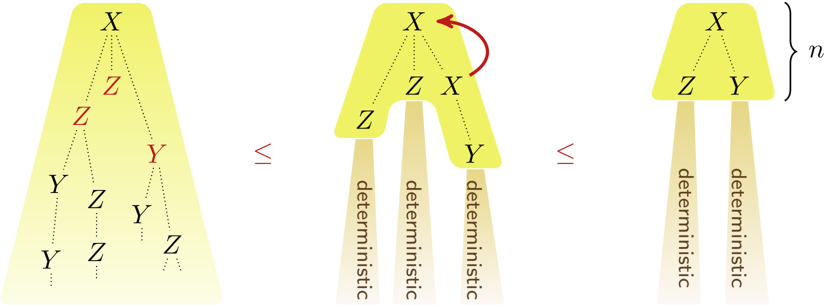
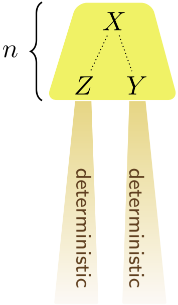


complicated tree 🌳 ultimately periodic nice tree 🌲

Computing Nice Trees

$$\mathrm{gfp}(\mathbf{F}) \;=\; \sum \left\{ \mathsf{yield}(🌲) \;\middle|\; \textsf{nice } 🌲 \right\} \;=\; \ldots$$



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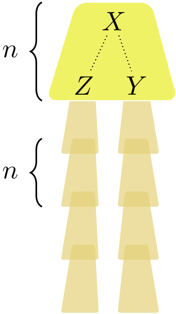

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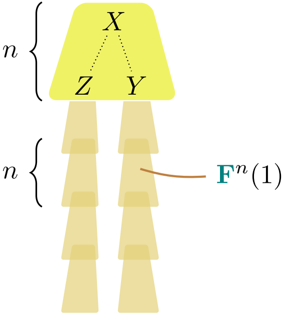

Computing Nice Trees

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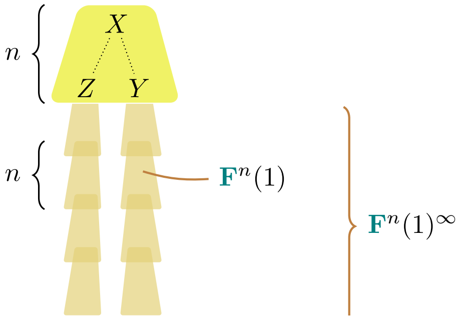

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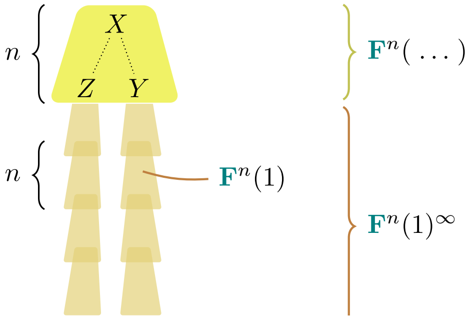

Summary

Result

- Greatest solutions of PES in absorptive, fully-continuous semirings ...
- ...are computable in a polynomial number of semiring operations

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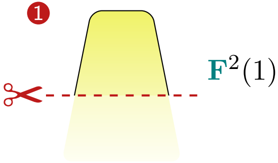

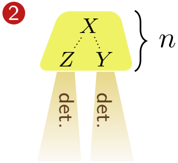

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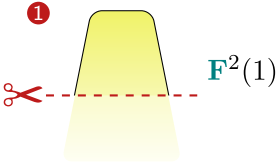

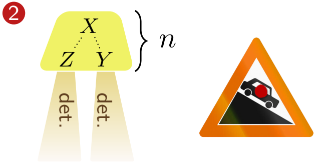

Alternative: Symbolic approach for $\mathbb{S}^\infty[\mathbf{X}]$

- Solve first equation for X, substitute and solve recursively

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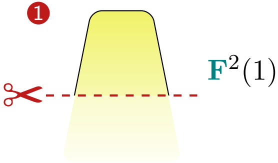

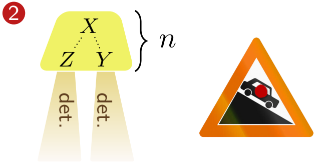

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Future Work: Compute nested fixed points in absorptive semirings

Symbolic Approach for Absorptive Polynomials

$$X = aX^2 + bY$$
$$Y = cX + d$$

Theorem

Greatest solution of $X = P(X)$ is

$$P(0) + P'(1)^\infty$$

and this holds uniformly over all instantiations of additional variables.

based on [Hopkins, Kozen, LICS'99]

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Issue: $K[X, Y] = K[X][Y]$

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