Fixpoint Games

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Joint work with

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Solving one Fixpoint Equation

We are interested in techniques for solving (systems of) fixpoint equations over a complete lattice

One-equation case

Solve the equation E given as $\quad x =_\eta f(x)$
where

- $f: L \to L$ is a monotone function over a complete lattice $(L, \sqsubseteq)$
- $\eta \in \{\mu, \nu\}$, indicating whether we are interested in the least (μ) or greatest (ν) fixpoint

The solution of E is denoted by $sol(E)$

Applications in concurrency theory, model checking, program analysis

Solving one Fixpoint Equation

Solution techniques

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- Kleene iteration: whenever f is (co-)continuous
  - $\eta = \mu$ (least fixpoint): $sol(E) = \bigsqcup_{i \in \mathbb{N}} f^i(\bot)$
  - $\eta = \nu$ (greatest fixpoint): $sol(E) = \bigsqcap_{i \in \mathbb{N}} f^i(\top)$

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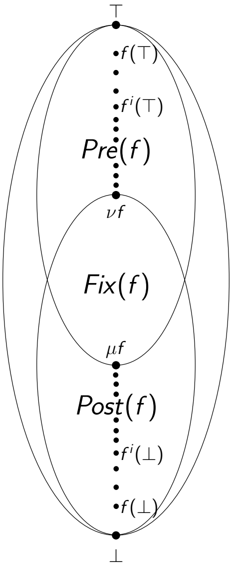
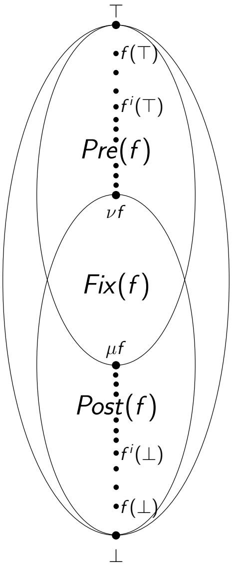
  - $\eta = \nu$ (greatest fixpoint): $sol(E) = \bigsqcap_{i \in \mathbb{N}} f^i(\top)$
- In order to check whether $l \sqsubseteq sol(E)$ for some $l \in L$:
  - $\eta = \mu$ (least fixpoint): use ranking functions
  - $\eta = \nu$ (greatest fixpoint): construct a postfix-point l' ($l' \sqsubseteq f(l')$) such that $l \sqsubseteq l'$

Solving one Fixpoint Equation

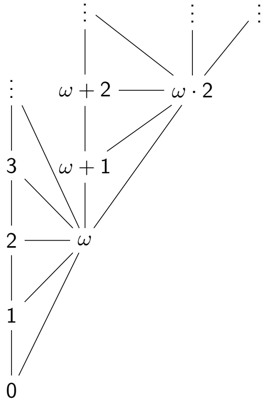


Solving one Fixpoint Equation



If f is *not* (co-)continuous:

$\rightsquigarrow$ Kleene iteration over the ordinals (beyond ω)



Solving one Fixpoint Equation

Examples

- Dataflow analysis (least or greatest fixpoint)
- Bisimilarity characterized as a greatest fixpoint
- Behavioural metric characterized a a least fixpoint
- ...

Solving (Systems of) Fixpoint Equations

System of fixpoint equations

Let L be a complete lattice. A system of equations E over L is of the following form, where $f_i \colon L^m \to L$ are monotone functions and $\eta_i \in \{\mu, \nu\}$.

$$\begin{array}{rcl} x_1 & =_{\eta_1} & f_1(x_1, \dots, x_m) \\ & \dots & \\ x_m & =_{\eta_m} & f_m(x_1, \dots, x_m) \end{array}$$

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The solution of E, denoted $sol(E) \in L^m$, is defined inductively as follows:

$$\begin{array}{rcl} sol(\emptyset) & = & () \\ sol(E) & = & (sol(E[x_m := s_m]), \, s_m) \end{array}$$

where $s_m = \eta_m(\lambda x.\, f_m(sol(E[x_m := x]), x))$

Solving (Systems of) Fixpoint Equations

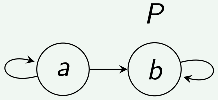
Remarks:

- $E[x_m := x]$ is a system of $m-1$ equations that one obtains by fixing the value of x_m as x and removing the last equation.
- Intuitively we fix the value of x_m as x, solve the remaining equation systems parameterized over x and then perform a fixpoint iteration (least or greatest) with f_m over x.
- The order of the equations matters.
- The solution is a fixpoint of the equation system (one of typically many fixpoints).

Solving (Systems of) Fixpoint Equations

Example: μ-calculus model checking

We consider the modal μ-calculus with $\Box$ ("all successor states satisfy ..."), $\Diamond$ ("some successor state satisfies ..."), least and greatest fixpoints.



$$\nu x_2.(\mu x_1.(\Diamond x_1 \lor (P \land \Diamond x_2)) \land \Box x_2)$$

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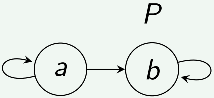

$$\nu x_2.(\mu x_1.(\Diamond x_1 \vee (P \wedge \Diamond x_2)) \wedge \Box x_2)$$

Equations over the powerset lattice of states:

$$\begin{aligned} x_1 &=_\mu \Diamond x_1 \cup (P \cap \Diamond x_2) \\ x_2 &=_\nu x_1 \cap \Box x_2 \end{aligned}$$

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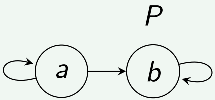

Equations over the powerset lattice of states:

$$x_1 \quad =_\mu \quad \Diamond x_1 \cup (P \cap \Diamond x_2)$$

x_1: "there exists a path such that eventually P holds and x_2 holds for some successor"

$$x_2 \quad =_\nu \quad x_1 \cap \Box x_2$$

x_2: "largest set such that x_1 holds and all successors satisfy x_2"

Combined: "from all reachable states there is a path along which P holds infinitely often"

Solving (Systems of) Fixpoint Equations

Efficient algorithms for μ-calculus model-checking

n: number of states d: alternation depth of formula

- Naive approach: use the definition $\rightsquigarrow O(n^d)$
- Reduce model-checking problem to a parity game and determine whether the existential player has a winning strategy
  - Local on-the fly algorithms [Stevens, Stirling] that perform an on-the fly search for a winning strategy of the existential player (proving that a given state satisfies a formula)
  - Progress measures [Jurdzinski] $\rightsquigarrow O(n^{\frac{d}{2}})$
  - Quasi-polynomial algorithms [Calude, Jain, Khoussainov, Bakhadyr, Li, Stephan] $\rightsquigarrow O(n^{\lceil \log d \rceil + c})$

Solving (Systems of) Fixpoint Equations

Example: lattice-valued μ-calculi

Variants: Non-boolean μ-calculi that do not check whether a formula holds in a state, but measure the "degree" with respect to which a formula is satisfied:

$$x \models \varphi \text{ is replaced by } [\![\varphi]\!]\colon X \to L$$

- Latticed μ-calculus [Kupferman, Lustig]
  $\leadsto$ over a lattice L
- Quantitative probabilistic μ-calculus [Huth, Kwiatkowska]
  $\leadsto$ over the real interval $L = [0, 1]$
- Łukasiewicz μ-calculus [Mio, Simpson]
  $\leadsto$ over the real interval $L = [0, 1]$

$\leadsto$ we require methods and techniques for solving fixpoint equations over general lattices (as opposed to powerset lattices)

Fixpoint Games

Aim: consider a game perspective for solving systems of fixpoint equations for general lattices

Let E be a system of m equations over a lattice L with basis B_L ($B_L \subseteq L$ such that every $l \in L$ can be obtained as $l = \bigsqcup B'$ where $B' \subseteq B_L$).

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- Given $b \in B_L$, $i \in \{1, \dots, m\}$ the existential player ($\exists$, Eve) wants to prove that $b \sqsubseteq s_i$.
- The universal player ($\forall$, Adam) is the adversary of $\exists$ and wants to show that $b \not\sqsubseteq s_i$.

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Precursor games:

- Parity games
- Unfolding games [Venema]
  - are being played on a powerset lattice
  - single fixpoint equation

Fixpoint Games

Fixpoint game (first version)

Position	Player	Moves
(b, i)	$\exists$	$(l_1, \ldots, l_m)$ such that $b \sqsubseteq f_i(l_1, \ldots, l_m)$
$(l_1, \ldots, l_m)$	$\forall$	(b', j) such that $b' \sqsubseteq l_j$

$b, b' \in B_L$, $\bot \not\in B_L$, $(l_1, \ldots, l_m) \in L^m$

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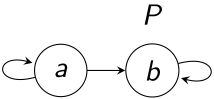
Winning condition ("parity condition")

	$\exists$	$\forall$
Finite game	$\forall$ unable to move	$\exists$ unable to move
Infinite game	$\eta_h = \nu$	$\eta_h = \mu$

Where $h \in \{1, \ldots, m\}$ is the highest equation index occurring infinitely often.

Fixpoint Games

We play the game on the powerset lattice $L = \mathcal{P}(\{a, b\})$ with basis $B_L = \{\{a\}, \{b\}\}$ for $b = \{a\}$, $i = 2$:



$$\begin{aligned} x_1 &=_\mu \Diamond x_1 \cup (P \cap \Diamond x_2) = f_1(x_1, x_2) \\ x_2 &=_\nu x_1 \cap \Box x_2 = f_2(x_1, x_2) \end{aligned}$$

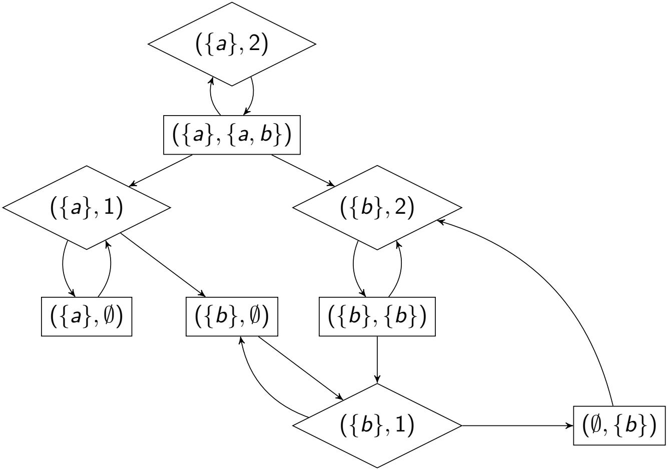
Remember: the second component of the solution contains all states such that "from all reachable states there is a path along which P holds infinitely often"

Fixpoint Games

Notation:

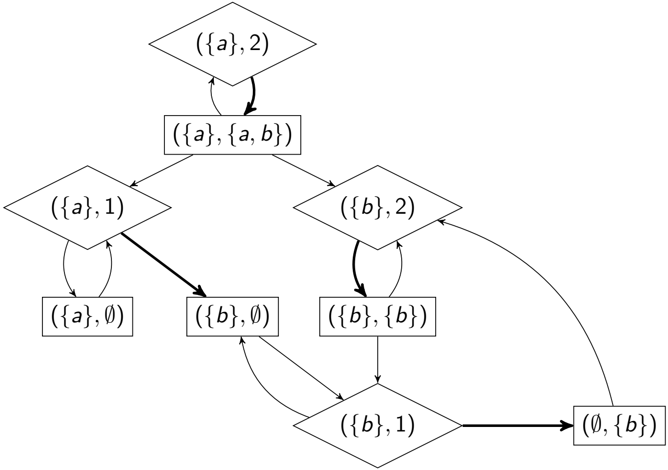
- Game positions (nodes) of $\exists$: $\Diamond$
- Game positions (nodes) of $\forall$: $\Box$

Fixpoint Games



Only minimal moves of $\exists$ are given.

Fixpoint Games



Thick arrows: winning strategy of $\exists$

Fixpoint Games

Is the game correct and complete for all (complete) lattices?
("$\exists$ has a winning strategy for $(b, i) \iff b \sqsubseteq s_i$")

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Counterexample

$L = \mathbb{N} \cup \{\omega\}$, $B_L = L \backslash \{0\}$
$f : L \to L$, $f(n) = n + 1$, $f(\omega) = \omega$

$$x \quad =_\mu \quad f(x)$$

We play a game to check whether ω is below the solution (= least fixpoint):

$$\omega \overset{\exists}{\leadsto} \omega \overset{\forall}{\leadsto} \omega \ldots$$

$\forall$ would win this game ... This means that something is wrong!

Fixpoint Games

In this case $\omega \sqsubseteq \bigsqcup_{i\in\mathbb{N}} f^i(0)$, but $\omega \not\sqsubseteq f^i(0)$ for all $i \in \mathbb{N}$.

However, in order to win, $\exists$ has to descend in the lattice in order to reach $\bot = 0$ and enforce a finite game. ($\exists$ has to be able to go below the "limit ordinals" in the fixpoint iteration.)

Solution: play with basis $B_L = \mathbb{N}\backslash\{0\}$. This forces $\forall$ to pick some $n \in \mathbb{N}$.

What are the restrictions on the basis in general?

Way-Below Relation, Algebraic and Continuous Lattices

Way-below relation (definition)

Given two elements $l, l' \in L$ we say that l is way-below l', written $l \ll l'$ when for all directed sets $D \subseteq L$, if $l' \sqsubseteq \bigsqcup D$ then there exists $d \in D$ such that $l \sqsubseteq d$.

- It holds that $\omega \not\ll \omega$, since $\omega \sqsubseteq \bigsqcup \mathbb{N}$, but ω is not below any element of the directed set $\mathbb{N}$.
- For two sets $Y, Y' \in \mathcal{P}(X)$ it holds that $Y \ll Y'$ iff $Y \subseteq Y'$ and Y finite.
- For $x, x' \in [0,1]$ it holds that $x \ll x'$ iff $x < x'$ or $x = 0$.

Way-Below Relation, Algebraic and Continuous Lattices

Algebraic lattice (definition)

An element $l \in L$ is compact if $l \ll l$.
A complete lattice L is algebraic if the compact elements form a basis.

- Every powerset lattice is algebraic.
- $\mathbb{N} \cup \{\omega\}$ is algebraic.
- $[0, 1]$ is *not* algebraic. (Only 0 is compact.)

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Soundness and completeness of the fixpoint game (first version)

The game is

- always correct ("$\exists$ has a winning strategy for $(b,i) \Rightarrow b \sqsubseteq s_i$")
- and complete ("$b \sqsubseteq s_i \Rightarrow \exists$ has a winning strategy for (b,i)") iff B_L consists of compact elements (and hence L is algebraic).

Way-Below Relation, Algebraic and Continuous Lattices

Continuous Lattice [Scott]

A complete lattice L is continuous if for all $l \in L$ it holds that $l = \bigsqcup\{l' \in L \mid l' \ll l\}$.

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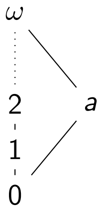
- Every algebraic lattice is continuous.
- $[0, 1]$ is a continuous lattice.
- The lattice to the right is not continuous: $a \not\ll a$, so $\bigsqcup\{l \in L \mid l \ll a\} = 0 \neq a$.



Way-Below Relation, Algebraic and Continuous Lattices

Fixpoint game (second version)

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$(l_1, \dots, l_m)$	$\forall$	(b', j) such that $b' \ll l_j$

$b, b' \in B_L$, $\bot \notin B_L$, $(l_1, \dots, l_m) \in L^m$

The winning conditions stay unchanged.

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The winning conditions stay unchanged.

Soundness and completeness of the fixpoint game (second version)

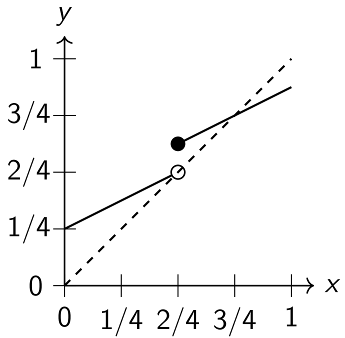
The game is

- always complete
- and correct iff L is continuous.

Way-Below Relation, Algebraic and Continuous Lattices

This works also for non-continuous functions! Let $f\colon [0,1] \to [0,1]$ be defined as:

$$f(x) = \begin{cases} \frac{1}{4} + \frac{1}{2}x & \text{if } 0 \le x < \frac{1}{2} \\ \frac{3}{8} + \frac{1}{2}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$



We have $\mu f = \frac{3}{4}$ (this fixpoint is not reachable in ω steps, since f is not continuous)

Game:

$\frac{3}{4} \overset{\exists}{\rightsquigarrow} \frac{3}{4} \overset{\forall}{\rightsquigarrow} \frac{3}{4} - \varepsilon \overset{\exists}{\rightsquigarrow} \frac{3}{4} - 2\varepsilon \rightsquigarrow \cdots \rightsquigarrow$
$\frac{5}{8} \overset{\exists}{\rightsquigarrow} \frac{1}{2} \overset{\forall}{\rightsquigarrow} \frac{1}{2} - \delta \rightsquigarrow \cdots \rightsquigarrow 0$
$\exists$ wins!

Conclusion

Further contributions

- Progress measures: computing the strategy of the existential player (global algorithm)
- Constraining the moves of the existential player via selections

$\rightsquigarrow$ POPL '19

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Further contributions

- Local algorithm for checking whether a lattice element is below the solution
- Integration with up-to techniques for stopping earlier
- Variant of the game: play on the powerset of the basis (sound and complete for all (complete) lattices)

$\rightsquigarrow$ CONCUR '20

Conclusion

Generalization of the quasipolynomial algorithms for parity games to finite lattices, based on fixpoint games:

Daniel Hausmann, Lutz Schröder: Quasipolynomial Computation of Nested Fixpoints. TACAS '21

Conclusion

Open questions

Does the theory developed here help to compute solutions of fixpoint equations over the reals, metrics and other infinite lattices?

- initial experiments with SMT solvers
- methods for approximating the solution