Relational Sums and Splittings in Categories of L-fuzzy Relations

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Motivation

1. When considering L-fuzzy relations, concepts such as relational sums and products usually add the additional requirement of the relations involved being crisp.
    1. Is this a restriction?
    2. And if so, how severe is this restriction?
2. The category Rel of regular binary relations has all splittings.
    1. Is this also true for L-fuzzy relations?
    2. Same questions as above when we split a crisp relation.

For the definition of Dedekind and arrow categories I refer to my presentations on Wednesday.

Relational Sums I

Definition

Suppose $\mathcal{R}$ is a Dedekind category. The relational sum of two objects A and B is an object $A + B$ together with two relations $\iota : A \to A + B$ and $\kappa : B \to A + B$ so that the following equations hold

$$\iota;\iota^{\smile} = \mathbb{I}_A, \quad \kappa;\kappa^{\smile} = \mathbb{I}_B, \quad \iota;\kappa^{\smile} = \perp\!\!\!\perp_{AB}, \quad \iota^{\smile};\iota \sqcup \kappa^{\smile};\kappa = \mathbb{I}_{A+B}.$$

$$\iota(a,c) = \begin{cases} 1 & \text{if } c = a \\ 0 & \text{otherwise} \end{cases}, \quad \kappa(b,c) = \begin{cases} 1 & \text{if } c = b \\ 0 & \text{otherwise} \end{cases}$$

Example

We will use the Boolean algebra $B_4 = \{0, a, b, 1\}$ with four elements as the lattice L.
The objects of $\mathcal{A}_1$ are the two sets $A = \{*\}$ and $B = \{*_1, *_2\}$.
Now, consider the relations $\iota, \kappa : A \to B$ defined by

$$\iota = \begin{array}{c} \\ * \end{array} \overset{\begin{array}{cc} *_1 & *_2 \end{array}}{\left[\begin{array}{cc} a & b \end{array} \right]}, \quad \kappa = \begin{array}{c} \\ * \end{array} \overset{\begin{array}{cc} *_1 & *_2 \end{array}}{\left[\begin{array}{cc} b & a \end{array} \right]}.$$

Then B together with ι, κ form a relational sum of A with itself, e.g.,

$$\iota;\iota^{\smile} = * \overset{\begin{array}{cc} *_1 & *_2 \end{array}}{\left[\begin{array}{cc} a & b \end{array} \right]}; \begin{array}{c} *_1 \\ *_2 \end{array} \overset{*}{\left[\begin{array}{c} a \\ b \end{array} \right]} = * \overset{*}{\left[\begin{array}{c} 1 \end{array} \right]} = \mathbb{I}_A$$

Define $\mathcal{A}_1$ to be the Dedekind category containing ι and κ. It turns out that $\mathcal{A}_1$ has a total of 26 relations split up as

$$|\mathcal{A}_1[A, A]| = 2, |\mathcal{A}_1[A, B]| = |\mathcal{A}_1[B, A]| = 4, |\mathcal{A}_1[B, B]| = 16.$$

In particular, we have $\mathcal{A}_1[A, B] = \{\perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}, \iota, \kappa\}$.

Relational Sums II

Theorem

Suppose $\mathcal{R}$ is a Dedekind category. Then the category of matrices $\mathcal{R}^+$ over $\mathcal{R}$ defined by

1. *The objects of $\mathcal{R}^+$ are pairs (f, I) where I is a non-empty set and f is a function from I to the objects of $\mathcal{R}$.*
2. *Given two objects (f, I) and (g, J) a morphism in $\mathcal{R}^+$ from (f, I) to (g, J) is a function Q from $I \times J$ to the relations in $\mathcal{R}$ so that $Q(i, j) : f(i) \to g(j)$.*

is a Dedekind category. Furthermore, the functor $E : \mathcal{R} \to \mathcal{R}^+$ defined by $E(A) = (A, \{*\})$ for objects and $E(Q)(*, *) = Q$ is a full embedding of Dedekind categories.

Relational Sums III

Theorem

Suppose $\mathcal{A}$ is an arrow category. Then $\mathcal{A}^+$ together with the operations $Q^{\downarrow}(i,j) = Q(i,j)^{\downarrow}$ and $Q^{\uparrow}(i,j) = Q(i,j)^{\uparrow}$ for $Q : (f,I) \to (g,J)$ is an arrow category. Furthermore, E is a full embedding of arrow categories.

Relational Sums III

Theorem

Suppose $\mathcal{A}$ is an arrow category. Then $\mathcal{A}^+$ together with the operations $Q^\downarrow(i,j) = Q(i,j)^\downarrow$ and $Q^\uparrow(i,j) = Q(i,j)^\uparrow$ for $Q : (f,I) \to (g,J)$ is an arrow category. Furthermore, E is a full embedding of arrow categories.

Theorem

Suppose $\mathcal{A}$ is an arrow category. Then $\mathcal{A}^+$ has crisp relational sums for all pairs of objects.

Splittings I

Suppose $Q : A \to A$ is crisp partial equivalence relation, i.e., Q is crisp, symmetric ($Q^{\smile} = Q$), and idempotent ($Q;Q = Q$).

Intuitively the object of the splitting representing the set of (existing) equivalence classes of Q.

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Definition

Suppose $\mathcal{R}$ is a Dedekind category, and $Q : A \to A$ is a partial equivalence relation, i.e., we have $Q;Q = Q$ and $Q^{\smile} = Q$. The splitting of Q is an object B together with a relation $R : B \to A$ so that

$$R;R^{\smile} = \mathbb{I}_B, \quad R^{\smile};R = Q.$$

Splittings II

Lemma

Suppose $\mathcal{A}$ is an arrow category, and $\alpha : A \to A$ is an ideal relation, i.e., $\pi_{AA};\alpha;\pi_{AA} = \alpha$. If a relation $R : B \to A$ splits α, then $\alpha = \pi_{AA}$.

Splittings III

Definition

Let $\mathcal{A}$ be an arrow category. A relation $Q : A \to B$ is called pseudo crisp (p-crisp) iff there are relations $R : A' \to A$ and $S' : B' \to B$ with $R^{\smile}; R = Q; Q^{\smile}$ and $S^{\smile}; S = Q^{\smile}; Q$ so that $R; Q; S^{\smile}$ is crisp.

Splittings III

Definition

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Lemma

Let $\mathcal{A}$ be an arrow category. Then we have:

1. *Every crisp relation is p-crisp.*
2. *If $Q : A \to A$ is a partial equivalence relation, then Q is p-crisp iff there is a relation $R : A' \to A$ with $R^{\smile}; R = Q$ so that $R; Q; R^{\smile}$ is crisp.*
3. *If $Q : A \to A$ is a partial equivalence relation that has a splitting, then Q is p-crisp.*

Splittings IV

Lemma

Let $\mathcal{A}$ be an arrow category. Then the following are equivalent:

1. *Every crisp partial equivalence splits in $\mathcal{A}$.*
2. *Every p-crisp partial equivalence splits in $\mathcal{A}$.*

Example I

The objects of $\mathcal{A}_2$ are the two sets $A = \{1, 2, 3, 4\}$ and $B = \{x, y\}$.
Consider the crisp partial equivalence relation $Q : A \to A$ and the relations $R_1, R_2 : B \to A$ defined in matrix form by

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad R_1 = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} a & a & b & 0 \\ b & b & a & 0 \end{bmatrix} \end{array}, \quad R_2 = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} b & b & a & 0 \\ a & a & b & 0 \end{bmatrix} \end{array}.$$

R_1 and R_2 both are a splitting of Q. For example,

$$R_1^{\smile}; R_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} a & b \\ a & b \\ b & a \\ 0 & 0 \end{bmatrix} \end{array}; \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} a & a & b & 0 \\ b & b & a & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = Q.$$

Example II

Now, we define $\mathcal{A}_2$ to be the smallest structure containing Q, R_1 and R_2. It turns out that $\mathcal{A}_2$ has a total of 1168 relations split up as

$$|\mathcal{A}_2[A,A]| = 1024, |\mathcal{A}_2[A,B]| = |\mathcal{A}_2[B,A]| = 64, |\mathcal{A}_2[B,B]| = 16.$$

Among the 64 relations with source B and target A the relation R_1 and R_2 from above are the only relation that split Q.

Splittings V

Theorem

Suppose $\mathcal{R}$ is a Dedekind category and $\mathcal{E}$ a class of partial equivalence relations from $\mathcal{R}$. Then the category $\mathcal{R}_\mathcal{E}$ is defined by

1. *The objects of $\mathcal{R}_\mathcal{E}$ are the elements of $\mathcal{E}$,*
2. *Given two objects $Q_1 : A \to A$ and $Q_2 : B \to B$ a morphism in $\mathcal{R}_\mathcal{E}$ from Q_1 to Q_2 is a relation $R : A \to B$ so that $Q_1; R; Q_2 = R$,*

is a Dedekind category. Furthermore, if $\mathcal{E}$ contains all identities of $\mathcal{R}$, then the functor $E : \mathcal{R} \to \mathcal{R}_\mathcal{E}$ defined by $E(A) = \mathbb{I}_A$ for objects and $E(R) = R$ is a full embedding of Dedekind categories.

Splittings VI

Theorem

Suppose $\mathcal{A}$ is an arrow category and $\mathcal{E}$ a class of crisp partial equivalence relations from $\mathcal{A}$. Then $\mathcal{A}_{\mathcal{E}}$ togther with the arrow operations inherited from $\mathcal{A}$ is an arrow category. Furthermore, if $\mathcal{E}$ contains all identities of $\mathcal{A}$, then E is a full embedding of arrow categories.

Splittings VI

Theorem

Suppose $\mathcal{A}$ is an arrow category and $\mathcal{E}$ a class of crisp partial equivalence relations from $\mathcal{A}$. Then $\mathcal{A}_{\mathcal{E}}$ togther with the arrow operations inherited from $\mathcal{A}$ is an arrow category. Furthermore, if $\mathcal{E}$ contains all identities of $\mathcal{A}$, then E is a full embedding of arrow categories.

Theorem

Suppose $\mathcal{A}$ is an arrow category and $\mathcal{E}$ a class of crisp partial equivalence relations from $\mathcal{A}$. If Q is a crisp partial equivalence relation in $\mathcal{A}$ so that Q is in $\mathcal{E}$, then Q has a crisp splitting in $\mathcal{A}_{\mathcal{E}}$.

Future work

1. Is there an algebraic characterization of p-crisp relations?
2. Similar investigation for relational products.
3. The definition of a relational power in arrow categories is different than for Dedekind categories. However, one may investigate the following modified questions:
    1. Do relational powers (Dedekind version) exists in arrow categories?
    2. And if so, how are the two versions related?

Thank you for your attention.