Greibach Normal Form and Simple Automata for Weighted ω-Context-Free Languages

Complete Star-Omega Semirings

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Motivation

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weighted ω-context-free languages

Introduction

to Weighted Languages

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Examples

- counting
- optimization (costs or gains)
- probabilities
- transducer
- average, discounting

Introduction: Weighted Automata (on Semirings)

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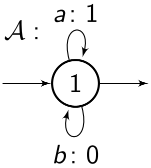

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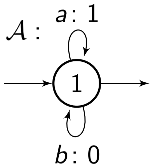

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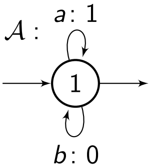

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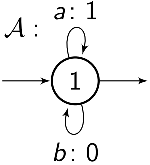

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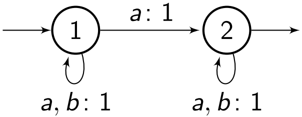

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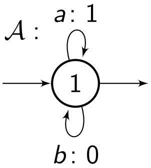

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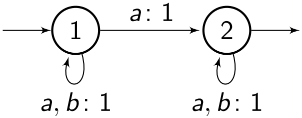

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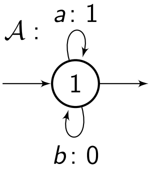

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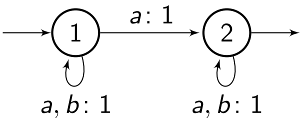

Examples

- unweighted (Boolean semiring): $(\mathbb{B}, \vee, \wedge, \perp, \top)$
- probabilities: $(\mathbb{Q}_+, +, \cdot, 0, 1)$
- transducer: $(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})$
- Viterbi: $([0, 1], \max, \cdot, 0, 1)$

Complete Semirings

Weight Structure

Complete and Continuous Star-Omega Semirings

Complete: "has infinite sums and infinite products"
Continuous: existence of certain fixpoints
Additional: star * and omega $^\omega$ operation

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Examples

$\langle \mathbb{R}_+^\infty, \min, +, \infty, 0 \rangle$, $\langle \mathbb{Q}_+^\infty, +, \cdot, 0, 1 \rangle$

Identities for Conway Semirings

Conway semiring:

- sum-star-equation: $(a+b)^* = (a^*b)^*a^*$
- product-star-equation: $(ab)^* = 1 + a(ba)^*b$
- it follows: $a^* = 1 + aa^*$ and $(ab)^*a = a(ba)^*$

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Lemma (Hebisch 1990)

Each complete star-omega semiring is a Conway semiring.

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$$\begin{aligned} Y_1 &\to Y_2 Y_1 \\ Y_2 &\to aY_2b \mid ab \end{aligned}$$

(with Büchi-accepting set $\{Y_1\}$) recognizes $\{a^n b^n \mid n \geq 1\}^\omega$.

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Algebraic Series

The canonical solutions are called *ω-algebraic series*.

Weighted ω-Pushdown Automata

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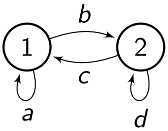

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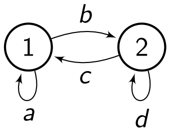

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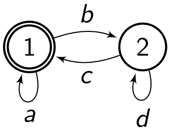

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$M^{\omega,k}$ contains infinite paths visiting the first k states infinitely often.

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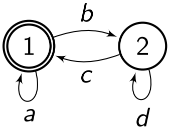

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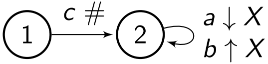
$$M^{\omega,1} = \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix}$$

Pushdown Matrix



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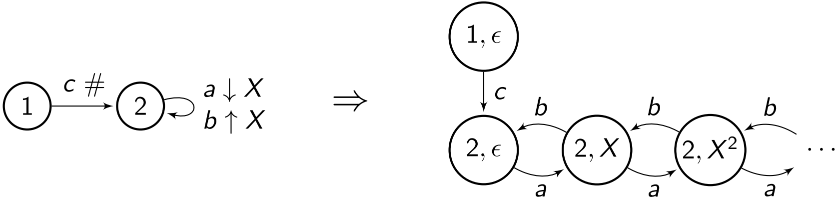

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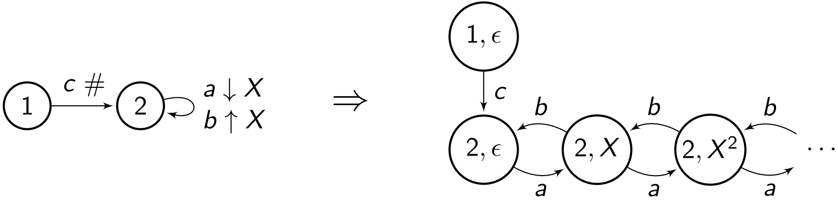

Pushdown Matrix

Intuition: *adjacency matrix* of the graph of configurations of a pushdown automaton

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Behavior:

$$\|\mathcal{A}\| = I(M^{\omega,k})_\epsilon$$

Back to the paper:

Results

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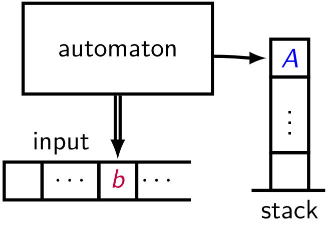
Thank you for your attention!

Backup

Notes

Unweighted Case

ω-Pushdown Automata



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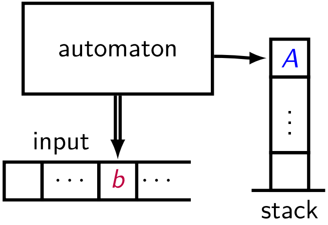

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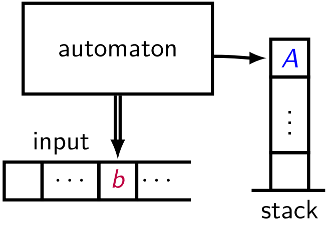

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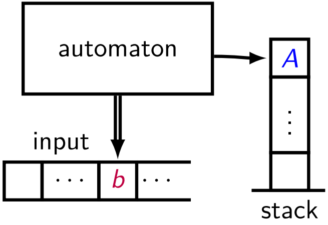

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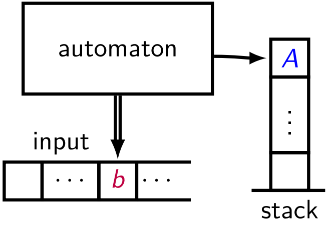

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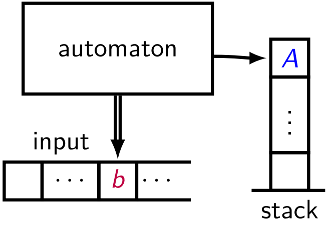

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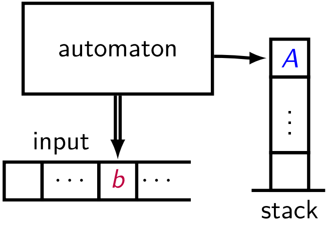

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Occurs in proof by Blass, Gurevich 2006.

Simple ω-Pushdown Automata accept all ω-Context-Free Languages

Construction: Simple ω-Pushdown Automata

Given ω-context-free grammar:

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$$X \to aXX \mid b$$

with nonterminals $\{S, X\}$
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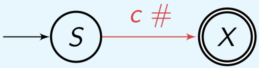

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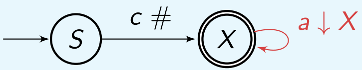

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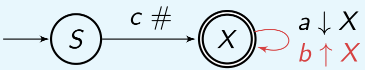

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Theorem

Let A be a continuous commutative star-omega semiring.

Let u be an ω-algebraic series over A.
Then there exists a weighted simple ω-pushdown automaton with behavior u.

Complete Semirings

Semiring $\langle A, +, \cdot, 0, 1\rangle$ is

- *complete* if it has infinitary sum operations $\sum_I : D^I \to D$, s.t.

$$(1) \quad \sum_{i\in\emptyset} d_i = 0, \quad \sum_{i\in\{k\}} d_i = d_k, \quad \sum_{i\in\{j,k\}} = d_j + d_k \text{ for } j \neq k,$$

$$(2) \quad \sum_{j\in J}\Big(\sum_{i\in I_j} d_i\Big) = \sum_{i\in I} d_i \quad \text{if } \bigcup_{j\in J} I_j = I \text{ and } I_j \cap I_k = \emptyset \text{ for } j \neq k,$$

$$(3) \quad \sum_{i\in I}(c\cdot a_i) = c \cdot \Big(\sum_{i\in I} a_i\Big), \quad \sum_{i\in I}(a_i \cdot c) = \Big(\sum_{i\in I} a_i\Big)\cdot c$$

- *complete starsemiring* if complete and *star* defined as

$$a^* = \sum_{j\geq 0} a^j$$

Continuous Semiring

A monoid $\langle A, +, 0 \rangle$ is

- *ordered* if commutative and
  partial ordering $\leq$ preserved by $+$ s.t. $0 \leq a$ for all $a \in A$
- *continuous* if ordered and
  each directed subset $D \subseteq A$ has $\sup D$ and
  Operation $+$ preserves the least upper bound of directed sets, i.e.,

$$a + \sup D = \sup(a + D)$$

Definition (Continuous semiring)

A semiring $\langle A, +, \cdot, 0, 1 \rangle$ is *continuous* if $\langle A, +, 0 \rangle$ is continuous monoid and multiplication is continuous, i.e.,

$$a \cdot \sup D = \sup(a \cdot D) \qquad \text{and} \qquad \sup D \cdot a = \sup(D \cdot a)$$

More Algebra - Infinite

The infinite product operation for complete semiring-semimodule pairs has these conditions:

$$\prod_{i\geq 1} s_i = \prod_{i\geq 1} (s_{n_{i-1}+1} \cdot \ldots \cdot s_{n_i})$$

$$s_1 \cdot \prod_{i\geq 1} s_{i+1} = \prod_{i\geq 1} s_i$$

$$\prod_{j\geq 1} \sum_{i_j \in I_j} s_{i_j} = \sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots} \prod_{j\geq 1} s_{i_j}$$

where in the first equation $0 = n_0 \geq n_1 \geq n_2 \geq \ldots$ and $I_1, I_2, \ldots$ are arbitrary index sets.

Canonical Solutions

Canonical Solutions of ω-Algebraic Systems

Mixed ω-Algebraic Systems

$$\begin{aligned} y_1 &= y_2 y_1 \\ y_2 &= a y_2 b + ab \end{aligned}$$

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Mixed ω-Algebraic Systems

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Write linear system of infinite words in matrix notation:

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The canonical solutions are called ω-algebraic series.

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Theorem

Let A be a continuous commutative star-omega semiring.

Let u be an ω-series over A. The following are equivalent:

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Proof Sketch

1. (above) is equivalent to
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$$s: \quad \begin{array}{c} S_1 \to \boldsymbol{\alpha_1} \\ \vdots \\ S_n \to \boldsymbol{\alpha_n} \end{array}$$

$\boldsymbol{\alpha_i}$, $\boldsymbol{\beta_i}$ in Greibach NF

$$t: \quad \begin{array}{c} T_1 \to \boldsymbol{\beta_1} \\ \vdots \\ T_m \to \boldsymbol{\beta_m} \end{array}$$

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$$u: \quad \begin{array}{l} U \to \boldsymbol{\alpha_1} \overline{U} \\ \overline{U} \to \boldsymbol{\beta_1} \overline{U} \end{array}$$

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$$u: \quad U \to \boldsymbol{\alpha_1}\overline{U} \qquad \overline{U} \to \boldsymbol{\beta_1}\overline{U}$$

But: u here not yet in quadratic Greibach NF!

Weighted ω-Pushdown Automata

Definition (Weighted ω-pushdown automaton)

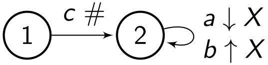
$\mathcal{A} = (n, \Gamma, I, M, k)$ with

- set of states $\{1, \ldots, n\}$, $n \geq 1$,
- finite pushdown alphabet Γ,
- initial state vector I,
- pushdown matrix $M \in \big((A\langle\Sigma \cup \{\epsilon\}\rangle)^{n \times n}\big)^{\Gamma^* \times \Gamma^*}$,
- integer k with $0 \leq k \leq n$. $(1, \ldots, k$ are Büchi-accepting states)

Behavior:

$$\|\mathcal{A}\| = I(M^{\omega,k})_\epsilon$$

Pushdown Matrix (Details)



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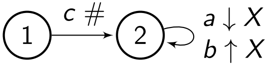
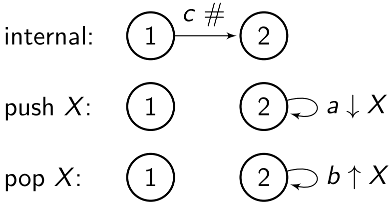


$\Rightarrow$



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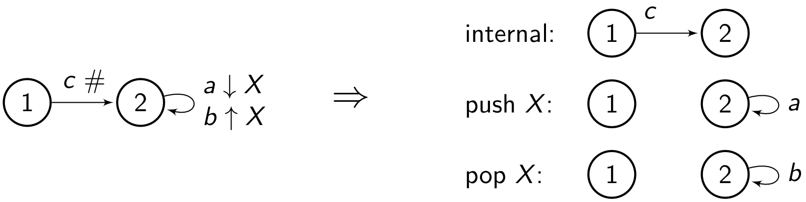

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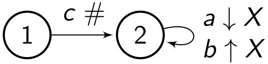
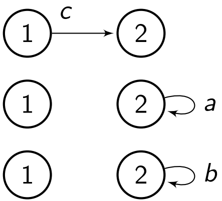


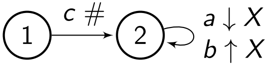
$\Rightarrow$

internal:, push X:, pop X:



$$\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

Pushdown Matrix (Details)



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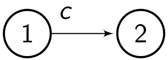
internal: 

$\begin{pmatrix} & c \\ & \end{pmatrix}$

push X: 

$\begin{pmatrix} & \\ & a \end{pmatrix}$

pop X: 

$\begin{pmatrix} & \\ & b \end{pmatrix}$

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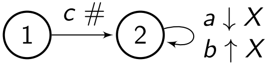

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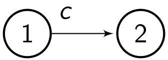
internal: 

$\begin{pmatrix} & c \\ & \end{pmatrix}$

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$$M = \begin{array}{c c} & \begin{array}{cc} \epsilon & X \end{array} \\ \begin{array}{c} \epsilon \\ X \end{array} & \begin{pmatrix} \begin{pmatrix} & c \\ & \end{pmatrix} & \begin{pmatrix} & \\ & a \end{pmatrix} \\ \begin{pmatrix} & \\ & b \end{pmatrix} & \begin{pmatrix} & c \\ & \end{pmatrix} \end{pmatrix} \end{array}$$

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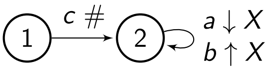

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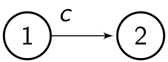
internal: 

$\begin{pmatrix} & c \\ & \end{pmatrix}$

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$\begin{pmatrix} & \\ & a \end{pmatrix}$

pop X: 

$\begin{pmatrix} & \\ & b \end{pmatrix}$

$$M = \begin{array}{c} \\ \epsilon \\ X \\ XX \\ XXX \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \epsilon & X & XX & XXX & \cdots \end{array} \\ \begin{pmatrix} \begin{pmatrix} & c \\ & \end{pmatrix} & \begin{pmatrix} & \\ & a \end{pmatrix} & 0 & 0 & \cdots \\ \begin{pmatrix} & \\ & b \end{pmatrix} & \begin{pmatrix} & c \\ & \end{pmatrix} & \begin{pmatrix} & \\ & a \end{pmatrix} & 0 & \cdots \\ 0 & \begin{pmatrix} & \\ & b \end{pmatrix} & \begin{pmatrix} & c \\ & \end{pmatrix} & \begin{pmatrix} & \\ & a \end{pmatrix} & \cdots \\ 0 & 0 & \begin{pmatrix} & \\ & b \end{pmatrix} & \begin{pmatrix} & c \\ & \end{pmatrix} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

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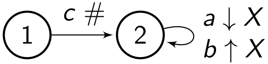

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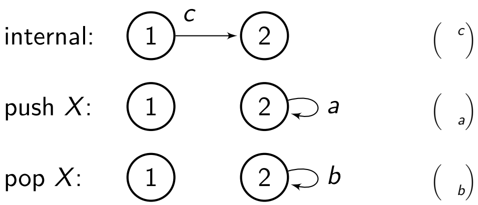
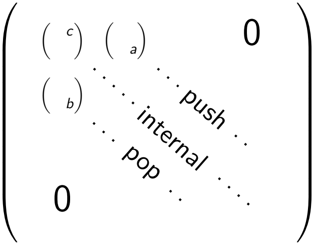


$M =$



(In)Finite Applications of a Matrix

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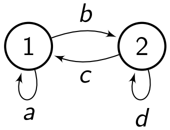

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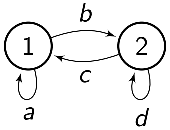

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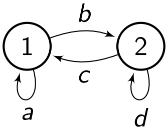

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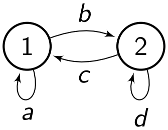

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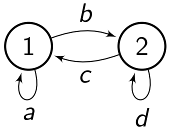

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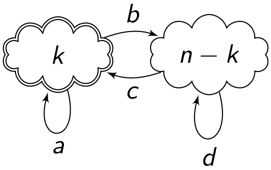
(In)Finite Applications of a Matrix - Büchi Condition

$$M = \begin{array}{cl} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{array}{l} \}\,k \\ \}\,n-k \end{array} \\ \underbrace{\qquad}_{k}\ \ \underbrace{\qquad}_{n-k} & \end{array}$$



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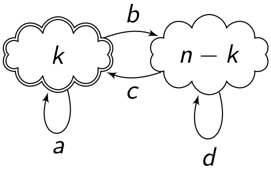

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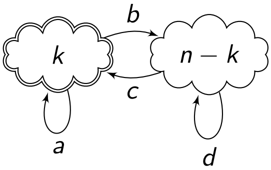

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Old Construction

Let r be the solution of a ω-algebraic system in Greibach NF

$$y_i = \sum_{1 \leq j,k \leq n} \sum_{a \in \Sigma} (p_i, a y_j y_k) a y_j y_k + \sum_{1 \leq j \leq n} \sum_{a \in \Sigma} (p_i, a y_j) a y_j + \sum_{a \in \Sigma} (p_i, a) a$$

for all $1 \leq i \leq n$.

From ω-Algebraic Systems to Automata

Theorem

Let A be a continuous and commutative star-omega semiring.

Let r be an ω-algebraic series.
Then there exists a weighted simple ω-pushdown automaton with behavior r.

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From ω-Algebraic Systems to Automata - Old Construction

Recall

$$y_i = \sum_{1 \leq j,k \leq n} \sum_{a \in \Sigma} (p_i, ay_jy_k)ay_jy_k + \sum_{1 \leq j \leq n} \sum_{a \in \Sigma} (p_i, ay_j)ay_j + \sum_{a \in \Sigma} (p_i, a)a; \quad 1 \leq i \leq n$$

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Old Construction (cont.)

We construct a weighted simple ω-pushdown automaton with n states:

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push: $\displaystyle\sum_{a\in\Sigma} (p_i, ay_jy_k)a \quad \Rightarrow \quad (i) \xrightarrow{a \downarrow y_k} (j)$

ignore: $\displaystyle\sum_{a\in\Sigma} (p_i, ay_j)a \quad \Rightarrow \quad (i) \xrightarrow{a \ \#} (j)$

pop: $\displaystyle\sum_{a\in\Sigma} (p_i, a)a \quad \Rightarrow \quad (i) \xrightarrow{a \uparrow y_k} (k)$