Finite Representation Property for Relation Algebra Reducts

Jaš Šemrl

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Introduction

The Conjecture

Failure of FRP with Negation and Composition

RA Embedding and FRP

Section 1

Introduction

Motivation

- Tarskian Representable Relation Algebras are badly behaved
- Decidability guarantees follow from FRP, FA
- FRP does not hold for the full language of Tarskian Relation Algebras
- Many negative FA results for reduct languages
- What about the FRP?

Proper Relation Algebras

Proper Relation Algebras are $\{0, 1, +, \cdot, -, 1', \smile, ;\}$-structures whose elements are binary relations over some base set X and

1. $0, 1, +, \cdot, -$ interpreted as proper Boolean operations
2. $1', \smile, ;$ interpreted as the relational identity, converse, and composition, respectively, i.e.

$$1' = \{(x, x) \mid x \in X\}$$
$$\breve{S} = \{(y, x) \mid (x, y) \in S\}$$
$$S; T = \{(x, z) \mid \exists y : (x, y) \in S, (y, z) \in T\}$$

for $S, T \subseteq X \times X$

Let τ be a RA-reduct language.

Finite Representation Property (FRP)

τ is said to have the Finite Representation Property (FRP) if all finite proper τ-structures are isomorphic to a proper τ-structure with a finite base.

Section 2

The Conjecture

Conjecture Statement

Let τ be a RA-reduct signature.

Conjecture

τ has the FRP if and only if

$$\{-, ;\} \not\subseteq \tau \not\supseteq \{\cdot, ;\}$$

Right to Left Implication

- All signatures containing $\{\cdot, ;\}$ have no FRP [Neu16]
- All signatures containing $\{-, ;\}$ have no FRP

Left to Right Implication

- Composition-free signatures have FRP
- Cayley Representation for Groups works for $\{;\}, \{1', ;\}$
- $\{\leq, ;\}$ have FRP [Zar59]
- $\{D, \smile, ;\} \subseteq \tau \subseteq \{0, 1, \leq, 1', \smile, ;\}$ have FRP [HE13, Šem21],
- $\{\leq, \backslash, /, ;\}$ have FRP [Rog20]

Section 3

Failure of FRP with Negation and Composition

Theorem

Any signature τ containing $\{-, ;\}$ fails to have the FRP.

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Proof:
The Point Algebra is the proper Relation Algebra over the base $\mathbb{Q}$ with the following 8 elements (interpreted arithmetically)

$$\{0, 1, =, \neq, <, \leq, >, \geq\}$$

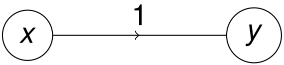
Observe how all operations in the language of RA are well defined for this algebra.

Assume there existed a proper τ-structure over a finite base X, isomorphic to the Point Algebra via some isomorphism θ.

There must exist $x, y \in X$ with $(x, y) \in 1^{\theta}$.

Claim 1

There must exist $x, y \in X$ with $(x, y) \in 1^\theta$.



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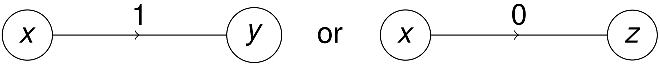

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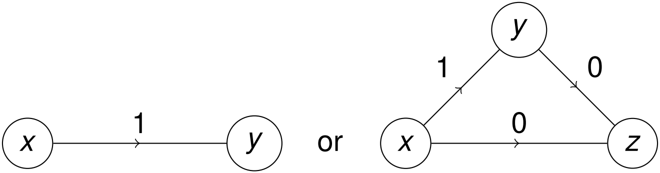

Claim 2

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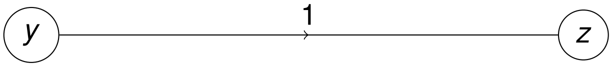
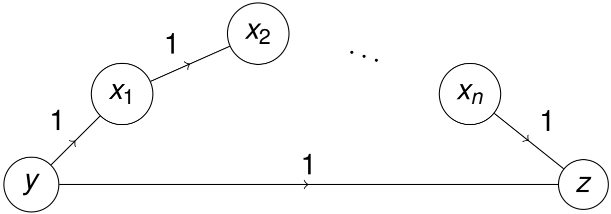


Claim 2

There must exist $x \in X$ with $(x, x) \in 1^{\theta}$.

$$1 = 1; 1 = 1; 1; 1 = \ldots = 1^{n+1} = ...$$



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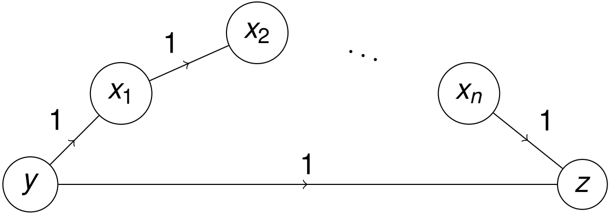

where $|X| = n - 1$.

So, there must exist $i < j$ such that $x_i = x_j = x$ and we get $(x, x) \in 1^\theta$ as $1 = 1^{j-i}$.

Claim 3

There must exist unique points $x_0, x_1, ..., x_m \in X$ for any $m < \omega$

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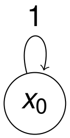

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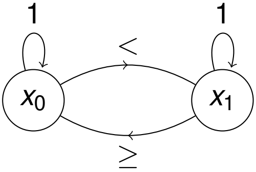

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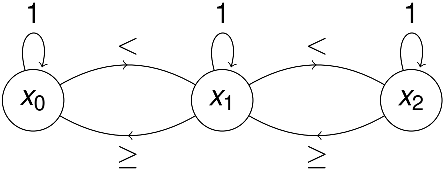

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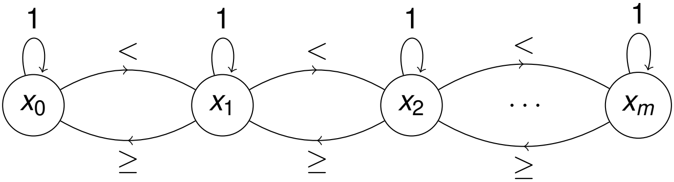

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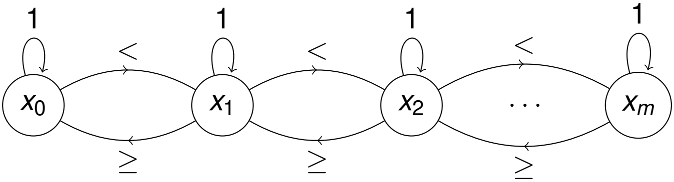

We've Reached a contradiction $\square$

Section 4

RA Embedding and FRP

Proposition

Every finite τ-structure, where $\{-, ;\} \not\subseteq \tau \not\supseteq \{\cdot, ;\}$, is finitely representable if and only if it embeds into a finite relation algebra.

Thank You!

ArXiv Identifier

http://arxiv.org/abs/2111.01213

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