Second-Order Properties of Undirected Graphs

Walter Guttmann
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1. Second-Order Properties
2. Orientations
3. Acyclic Graphs
4. Further Results

Relational and Logical Specifications

- transitive: $RR \subseteq R$

$$\forall x \forall y \forall z((x,y) \in R \ \wedge \ (y,z) \in R \ \Rightarrow \ (x,z) \in R)$$

- reachable: $P \subseteq R^* Q$

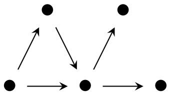
$$\exists n \exists x_1 \ldots \exists x_{n-1} \forall i (0 \leq i < n \ \Rightarrow \ (x_i, x_{i+1}) \in R)$$

- acyclic: $R^+ \subseteq \bar{\mathsf{I}}$

$$\neg \exists n \exists x_0 \ldots \exists x_n (\forall i (0 \leq i < n \ \Rightarrow \ (x_i, x_{i+1}) \in R) \ \wedge \ (x_n, x_0) \in R)$$

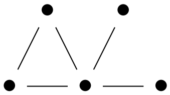
Acyclic Graphs

directed acyclic



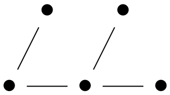
$R^+ \subseteq \bar{\mathsf{I}}$

undirected



$R = R^\mathsf{T}$

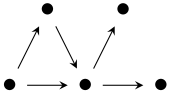
undirected acyclic

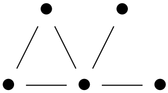


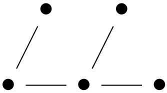
?

Acyclic Graphs

directed acyclic | undirected | undirected acyclic



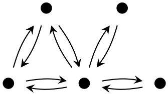


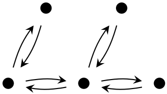


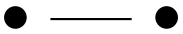
$R^+ \subseteq \bar{\mathsf{I}}$

$R = R^\mathsf{T}$

?







Quantifying over Relations

- pair-dense

$$\forall R(0 \neq R \subseteq I \;\Rightarrow\; \exists Q(0 \neq Q \subseteq R \;\wedge\; Q\overline{I}Q\overline{I}Q \subseteq I))$$

- intermediate point theorem

$$P \subseteq RSQ \;\Leftrightarrow\; \exists X(X \text{ is a point} \;\wedge\; P \subseteq RX \;\wedge\; X \subseteq SQ)$$

- axiom of choice

$$\forall R(I \subseteq RR^{\mathsf{T}} \;\Rightarrow\; \exists Q(Q \subseteq R \;\wedge\; Q^{\mathsf{T}}Q \subseteq I \subseteq QQ^{\mathsf{T}}))$$

Map Fusion

- point-wise

$$\text{map}\ f\ (\text{map}\ g\ xs) = \text{map}\ (f \circ g)\ xs$$

- point-free

$$\text{map}\ f \circ \text{map}\ g = \text{map}\ (f \circ g)$$

Map Fusion

- point-wise

$$\forall f \forall g \forall xs(\mathsf{map}\ f\ (\mathsf{map}\ g\ xs) = \mathsf{map}\ (f \circ g)\ xs)$$

- point-free?

$$\forall f \forall g(\mathsf{map}\ f \circ \mathsf{map}\ g = \mathsf{map}\ (f \circ g))$$

Map Fusion

- point-wise

$$\forall f \forall g \forall xs (\mathsf{map}\ f\ (\mathsf{map}\ g\ xs) = \mathsf{map}\ (f \circ g)\ xs)$$

- point-free?

$$\forall f \forall g (\mathsf{map}\ f \circ \mathsf{map}\ g = \mathsf{map}\ (f \circ g))$$

- really point-free

$$(\mathsf{flip}\ (\circ)\ \mathsf{map}) \circ (\circ) \circ \mathsf{map} = ((\circ)\ \mathsf{map}) \circ (\circ)$$

Research

- specify undirected acyclic graphs by quantifying over relations
- study such specifications using Kleene relation algebras

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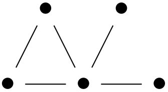
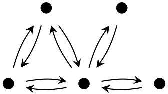
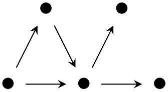
Orientations

undirected graph

pick a direction for each edge







Q is an orientation of $R \;\Leftrightarrow\; Q \cap Q^{\mathsf{T}} = 0 \;\wedge\; Q \cup Q^{\mathsf{T}} = R$

Orientability in Relation Algebras

R is orientable $\qquad \exists Q(Q \cap Q^{\mathsf{T}} = 0 \;\wedge\; Q \cup Q^{\mathsf{T}} = R)$

$$\Downarrow$$

R is symmetric and irreflexive $\qquad R = R^{\mathsf{T}} \;\wedge\; R \subseteq \bar{\mathsf{I}}$

Orientability in Relation Algebras

R is orientable $\qquad \exists Q(Q \cap Q^{\mathsf{T}} = 0 \;\wedge\; Q \cup Q^{\mathsf{T}} = R)$

$\Downarrow \qquad \not\Uparrow$ $\mathrm{Cm}(\mathbb{Z}_2)$

R is symmetric and irreflexive $\qquad R = R^{\mathsf{T}} \;\wedge\; R \subseteq \bar{\mathsf{I}}$

Specific Orientations

- tournament Q = orientation of $\bar{\mathsf{I}}$

$$Q \cap Q^\mathsf{T} = 0 \;\wedge\; Q \cup Q^\mathsf{T} = \bar{\mathsf{I}}$$

- pseudoforest R = injectively orientable

$$\exists Q(Q \cap Q^\mathsf{T} = 0 \;\wedge\; Q \cup Q^\mathsf{T} = R \;\wedge\; QQ^\mathsf{T} \subseteq \mathsf{I})$$

- comparability graph R = transitively orientable

$$\exists Q(Q \cap Q^\mathsf{T} = 0 \;\wedge\; Q \cup Q^\mathsf{T} = R \;\wedge\; QQ \subseteq Q)$$

- directed spanning forest Q of R

$$Q \subseteq R \subseteq (Q \cup Q^\mathsf{T})^* \;\wedge\; Q^+ \subseteq \bar{\mathsf{I}} \;\wedge\; QQ^\mathsf{T} \subseteq \mathsf{I}$$

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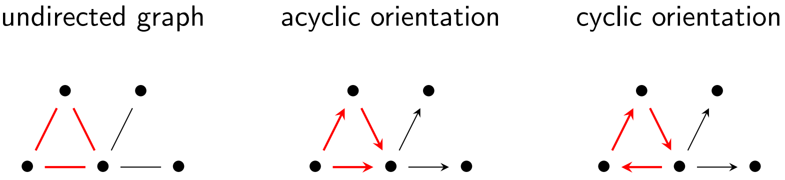
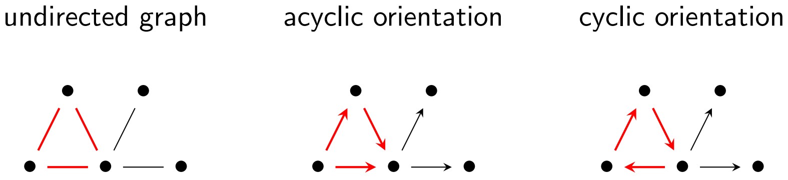
Acyclic 1/2



(A1) every orientation of R is acyclic:

$$\forall Q(Q \cap Q^{\mathsf{T}} = 0 \ \wedge \ Q \cup Q^{\mathsf{T}} = R \ \Rightarrow \ Q^{+} \subseteq \bar{\mathsf{I}})$$

Acyclic 1/2



(A1) every orientation of R is acyclic:

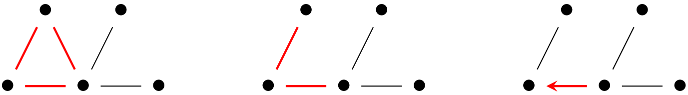
$$\forall Q(Q \cap Q^{\mathsf{T}} = 0 \;\wedge\; Q \cup Q^{\mathsf{T}} = R \;\Rightarrow\; Q^{+} \subseteq \bar{\mathsf{I}})$$

(A2) every asymmetric subset of R is acyclic:

$$\forall Q(Q \cap Q^{\mathsf{T}} = 0 \;\wedge\; Q \subseteq R \;\Rightarrow\; Q^{+} \subseteq \bar{\mathsf{I}})$$

Acyclic 3/4

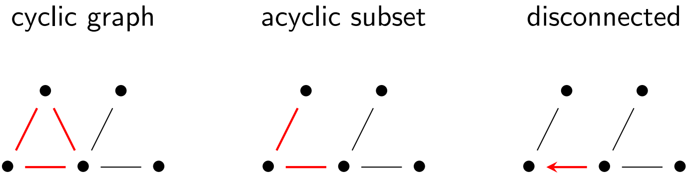
cyclic graph | acyclic subset | disconnected



(A3) R is minimal among graphs with the same connectivity:

$$\forall Q (Q \subseteq R \subseteq Q^* \Rightarrow Q = R)$$

Acyclic 3/4



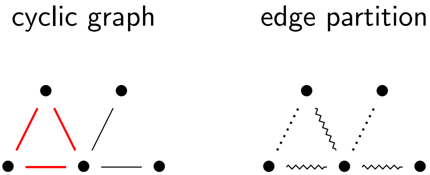
(A3) R is minimal among graphs with the same connectivity:

$$\forall Q(Q \subseteq R \subseteq Q^* \;\Rightarrow\; Q = R)$$

(A4) R contains no redundant connectivity information:

$$\forall Q(Q \subseteq R \;\Rightarrow\; R \cap Q^* \subseteq Q)$$

Acyclic 5

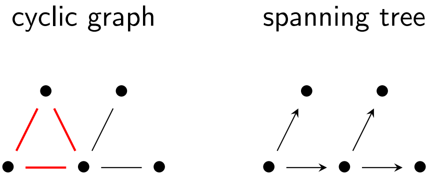


(A5) partitions of edges of R do not share paths:

$$\forall Q (Q \subseteq R \;\Rightarrow\; Q^* \cap (R \cap \overline{Q})^* = \mathsf{I})$$

$$\forall P \forall Q (P \cap Q = 0 \;\wedge\; P \cup Q = R \;\Rightarrow\; P^* \cap Q^* = \mathsf{I})$$

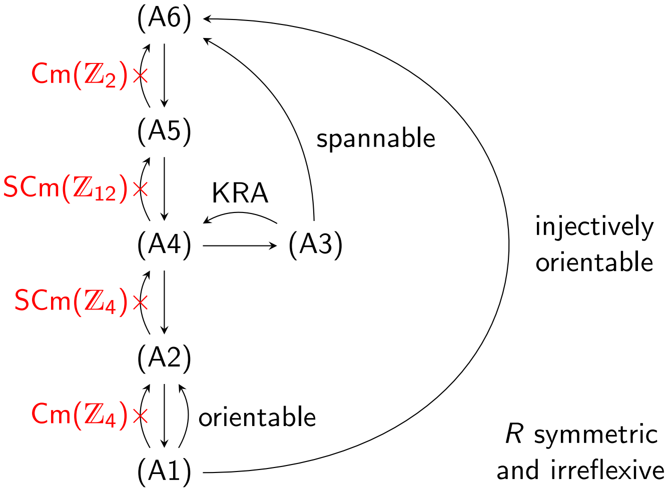
Acyclic 6



(A6) some orientation of R is a directed forest:

$$\exists Q(Q \cup Q^{\mathsf{T}} = R \ \wedge\ Q^{+} \subseteq \bar{\mathsf{I}} \ \wedge\ QQ^{\mathsf{T}} \subseteq \mathsf{I})$$

Connections in Kleene Relation Algebras



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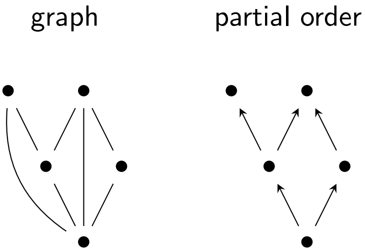
Transitive Orientations



$\bar{\mathsf{I}}$ is transitively orientable

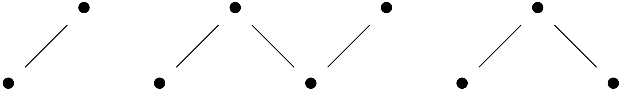
$$\Leftrightarrow \quad \exists Q(Q \cup Q^{\mathsf{T}} = \bar{\mathsf{I}} \ \wedge \ QQ \subseteq Q)$$

$$\Leftrightarrow \quad \exists Q(Q^{+} \subseteq \bar{\mathsf{I}} \ \wedge \ Q^{*} \cup Q^{*\mathsf{T}} = \top)$$

Further Properties

- all components of R are undirected simple paths

$$\exists Q(Q \cup Q^{\mathsf{T}} = R \ \wedge\ Q^{+} \subseteq \bar{\mathsf{I}} \ \wedge\ QQ^{\mathsf{T}} \subseteq \mathsf{I} \ \wedge\ Q^{\mathsf{T}}Q \subseteq \mathsf{I})$$



- reachable already in asymmetric subset

$$\forall P \forall Q(\mathrm{arc}\ P \ \wedge\ P \subseteq Q^{*} \ \Rightarrow \ \exists R(R \cap R^{\mathsf{T}} = 0 \ \wedge\ R \subseteq Q \ \wedge\ P \subseteq R^{*}))$$

- arc P

$$P \top P^{\mathsf{T}} \subseteq \mathsf{I} \ \wedge\ P^{\mathsf{T}} \top P \subseteq \mathsf{I} \ \wedge\ \top P \top = \top$$

Constructive Algorithmic Proofs

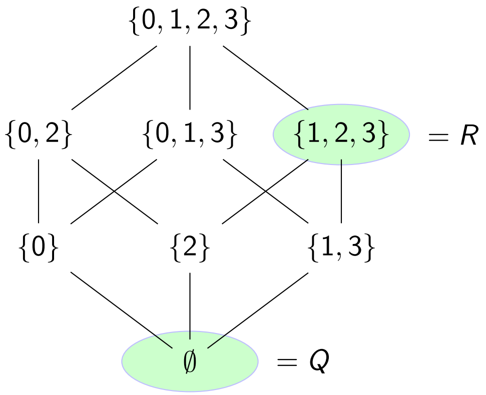
Assume finite algebra (for termination) and

$$\forall Q(Q \neq 0 \;\Rightarrow\; \exists P(\mathsf{arc}\ P \;\wedge\; P \subseteq Q))$$

Then:

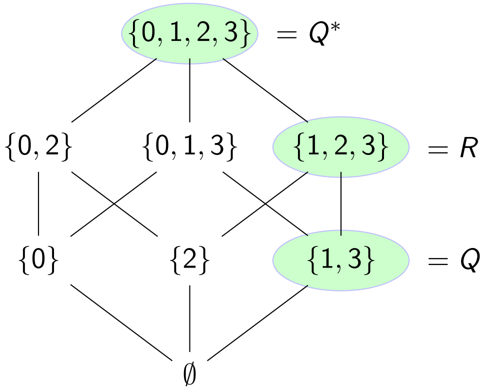
- $\bar{\mathsf{I}}$ is transitively orientable (Szpilrajn's algorithm)
- every symmetric irreflexive R is orientable (corollary)
- every symmetric R is spannable (Kruskal's algorithm)
- reachable already in asymmetric subset (breadth-first search)

Counterexample in SCm($\mathbb{Z}_4$)



(A2) $\forall Q(Q \cap Q^\mathsf{T} = 0 \ \wedge \ Q \subseteq R \ \Rightarrow \ Q^+ \subseteq \bar{\mathsf{I}})$ holds

Counterexample in $\mathrm{SCm}(\mathbb{Z}_4)$



(A2) $\forall Q (Q \cap Q^\mathsf{T} = 0 \;\wedge\; Q \subseteq R \;\Rightarrow\; Q^+ \subseteq \bar{\mathsf{I}})$ holds

(A3) $\forall Q (Q \subseteq R \subseteq Q^* \;\Rightarrow\; Q = R)$ fails

Conclusion

- all results proved in Isabelle/HOL
- in Stone-Kleene relation algebras for weighted graphs
- use new specifications for verifying graph algorithms
- formalise further graph properties