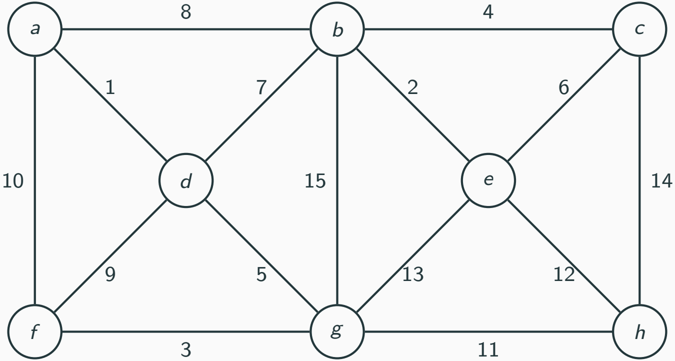
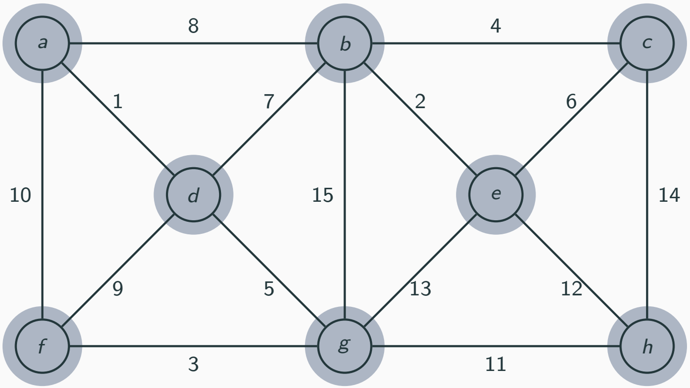
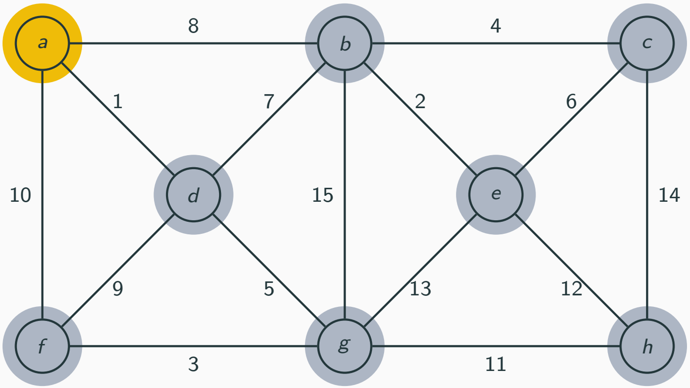
Relation-algebraic Verification of Borůvka's Minimum Spanning Tree Algorithm

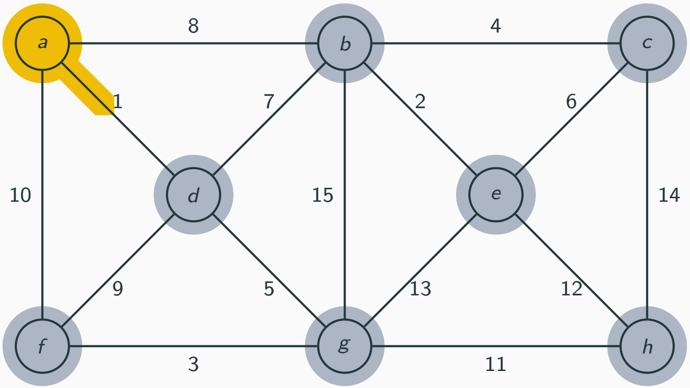
Nicolas Robinson-O'Brien   Walter Guttmann

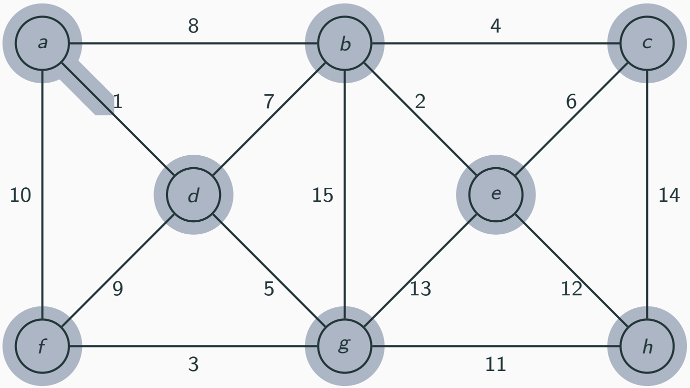
University of Canterbury

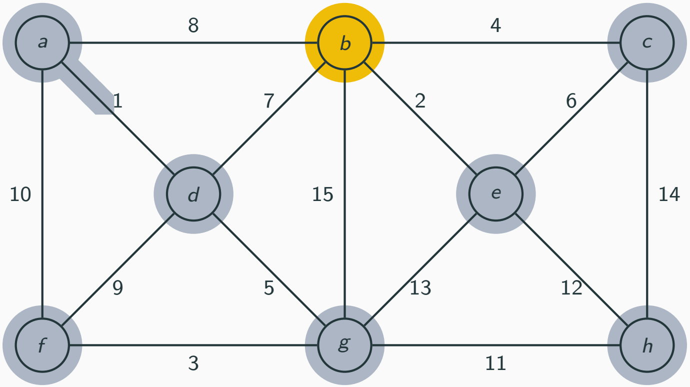

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

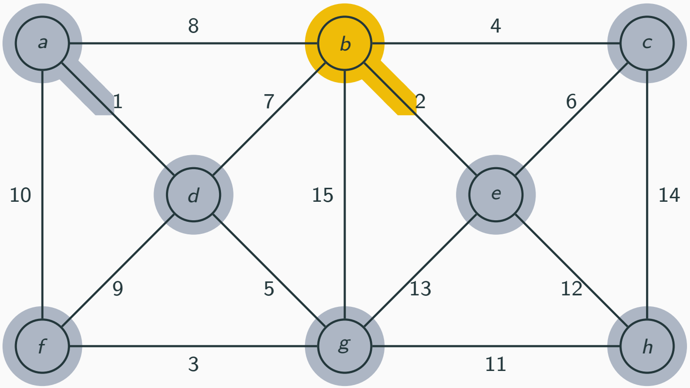

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

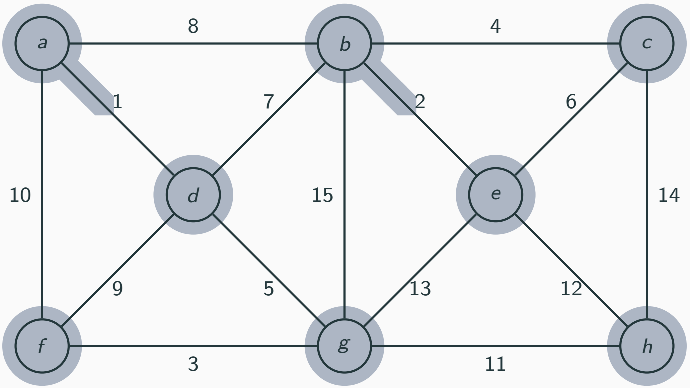

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

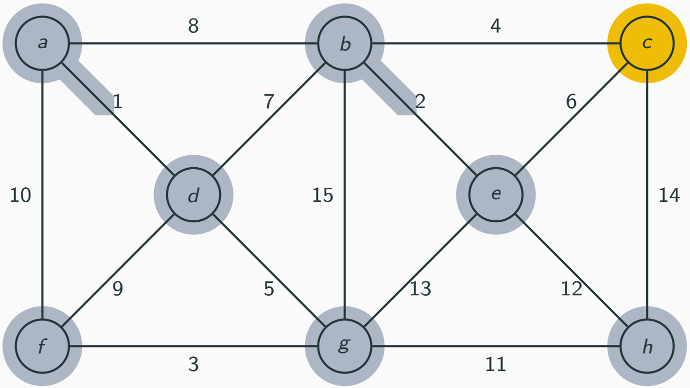

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

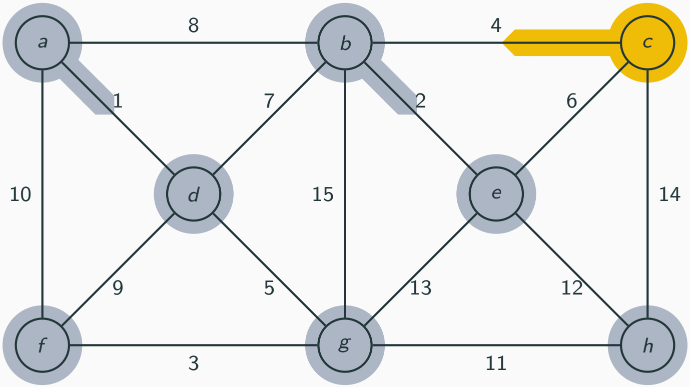

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

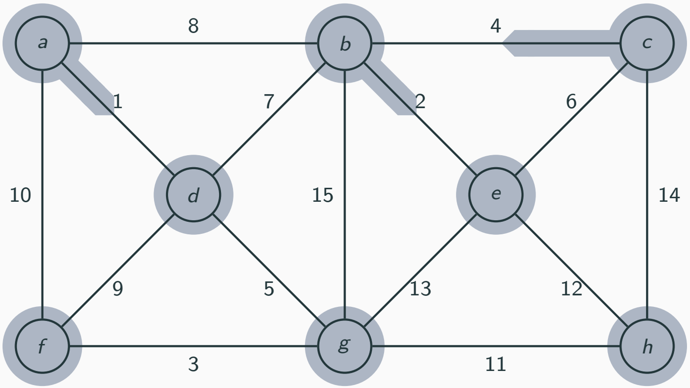

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

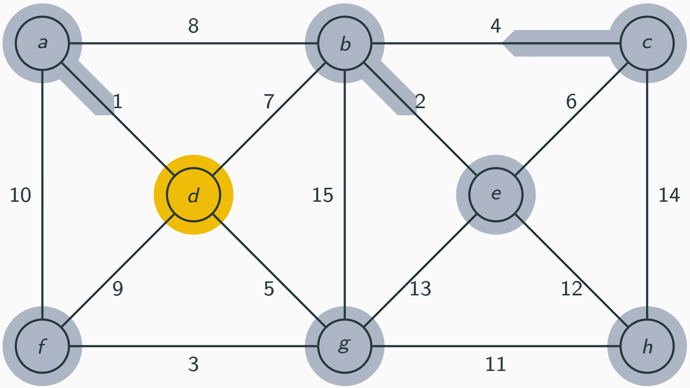

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

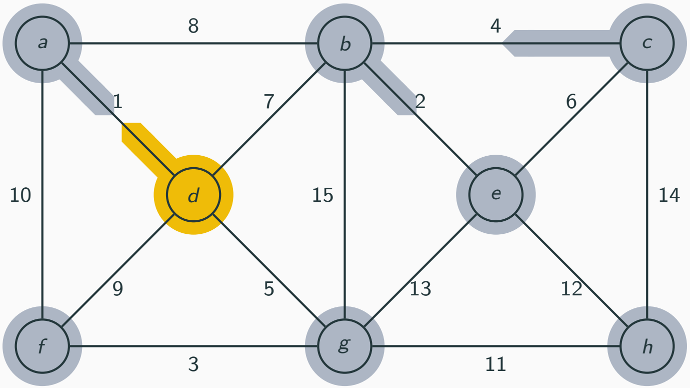

a
8
b
4
c
1
7
2
6
10
d
15
e
14
9
5
13
12
f
g
h
3
11

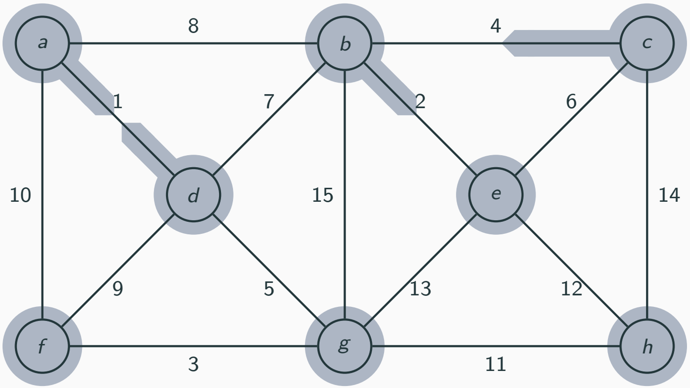

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11


a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

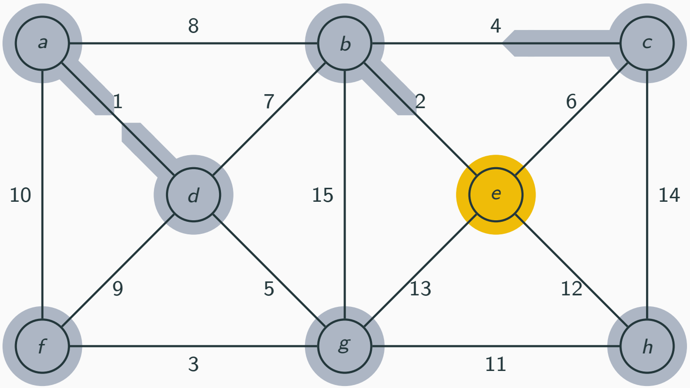

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

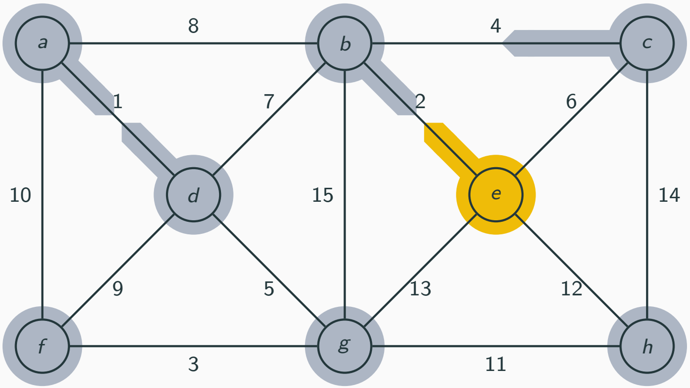

a
8
b
4
c
1
7
2
6
10
d
15
e
14
9
5
13
12
f
3
g
11
h

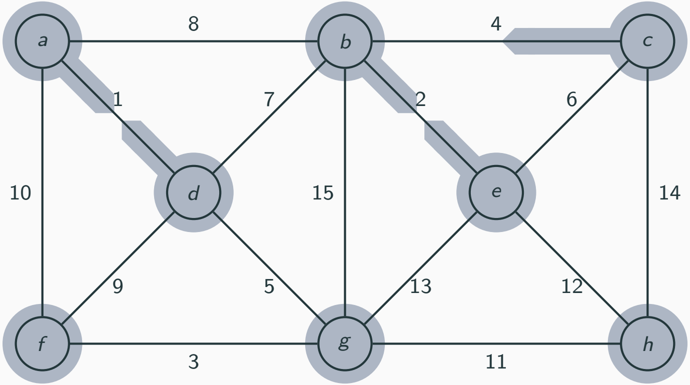

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

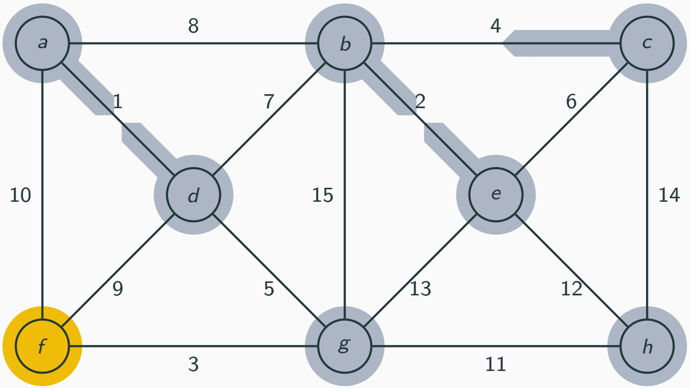


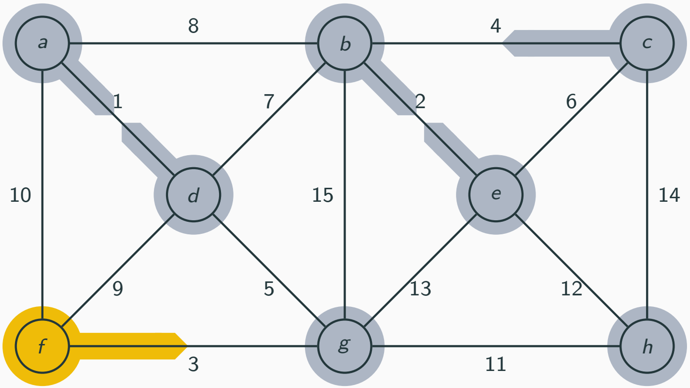
a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

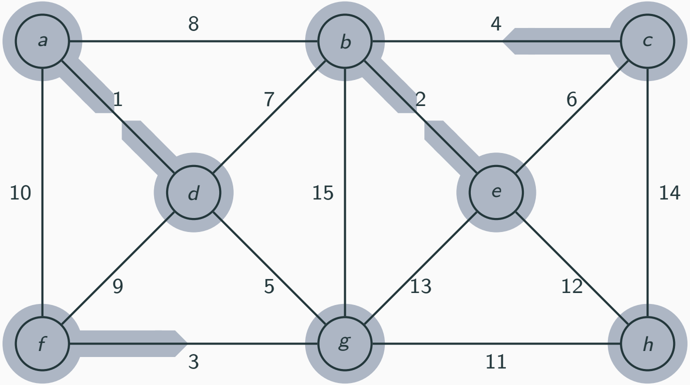

8
4
a
b
c
1
7
2
6
10
d
15
e
14
9
5
13
12
f
g
h
3
11

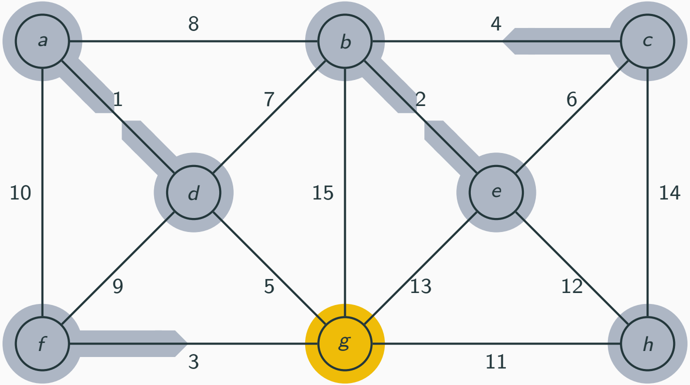

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

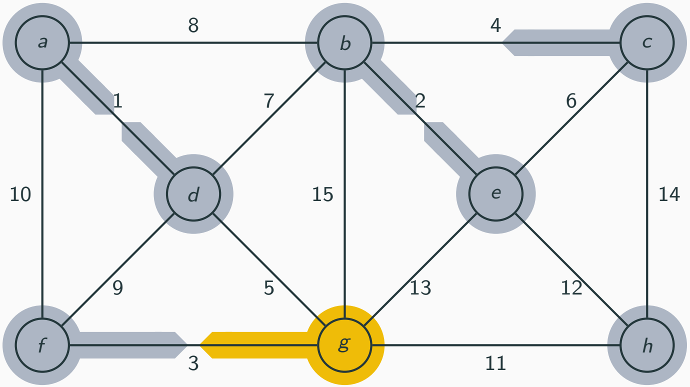

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

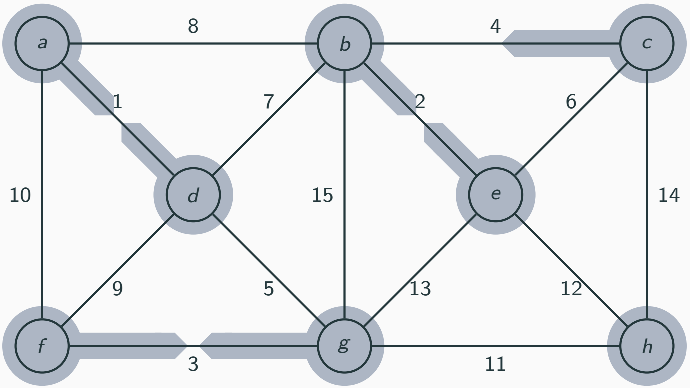

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

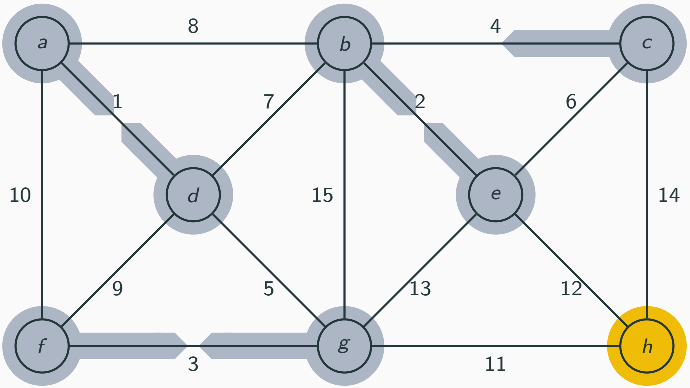

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

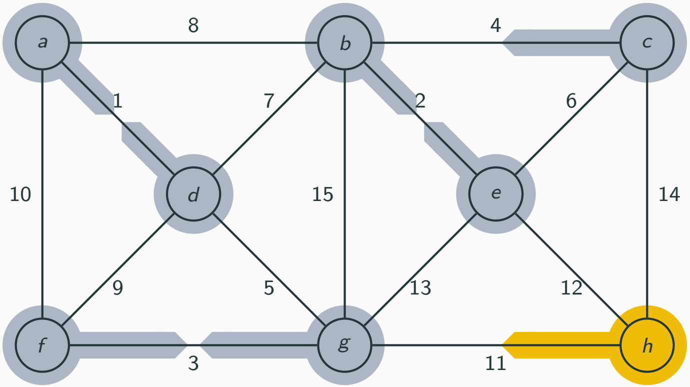

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11


a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

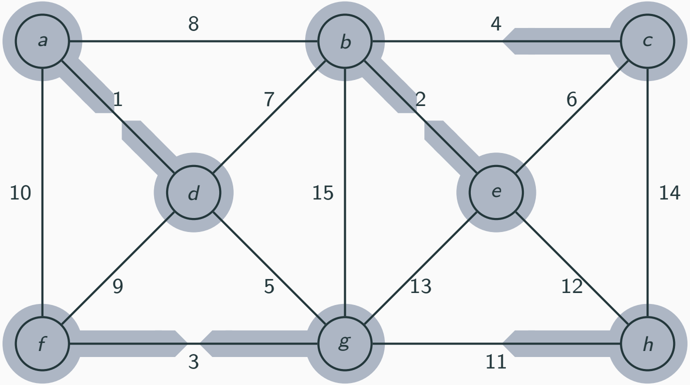

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

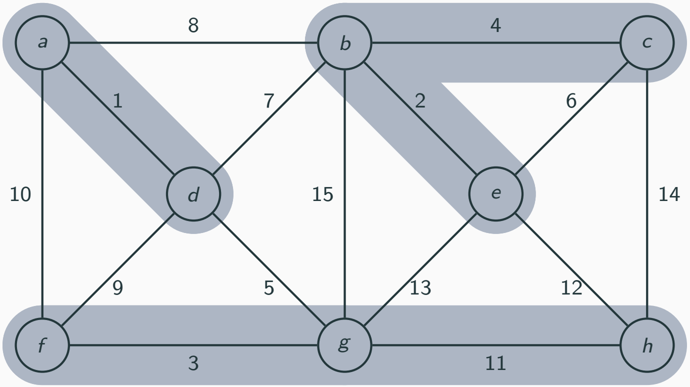

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

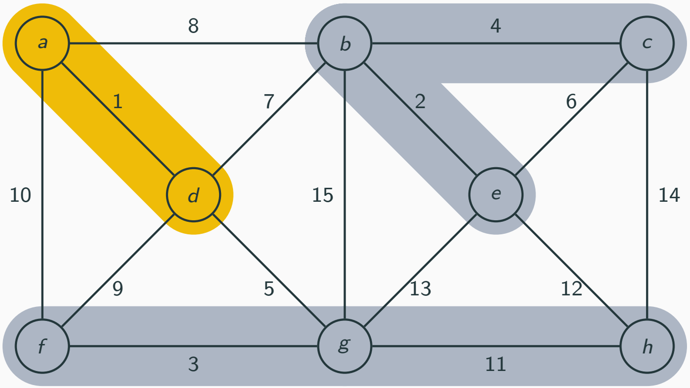

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

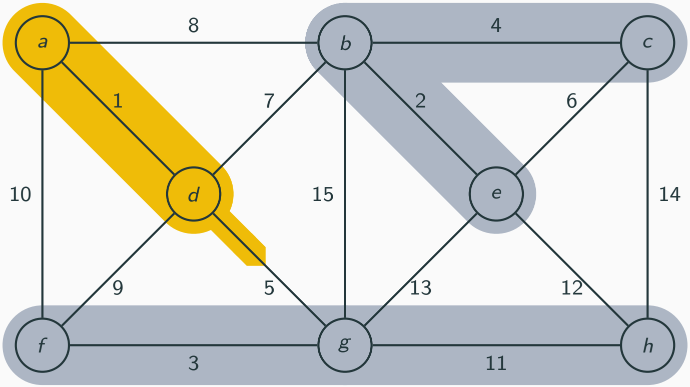


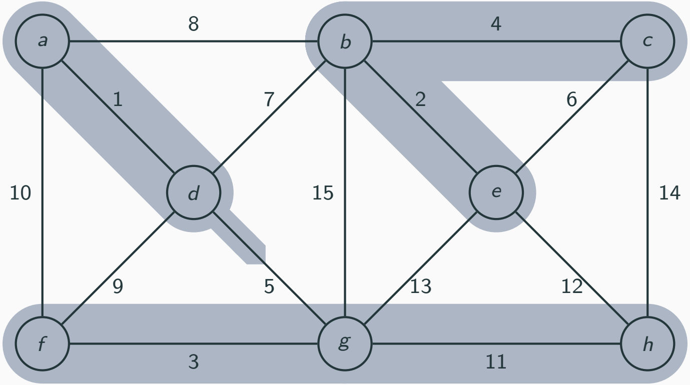
a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

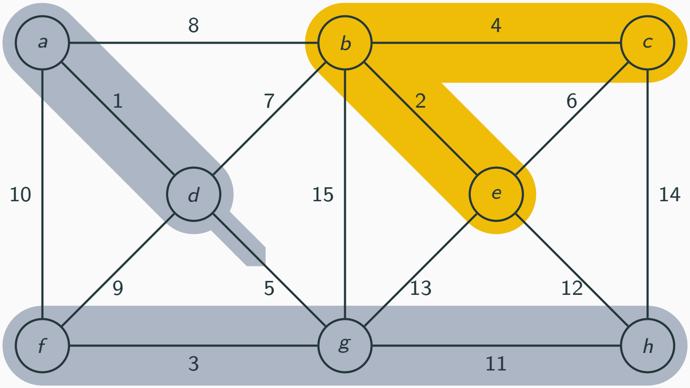

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

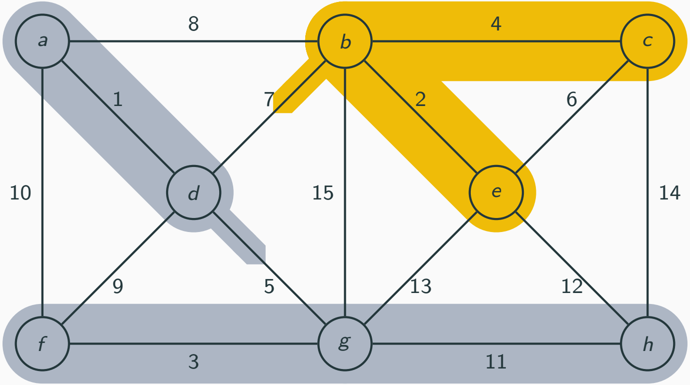

8
4
a
b
c
1
7
2
6
10
d
15
e
14
9
5
13
12
f
g
h
3
11

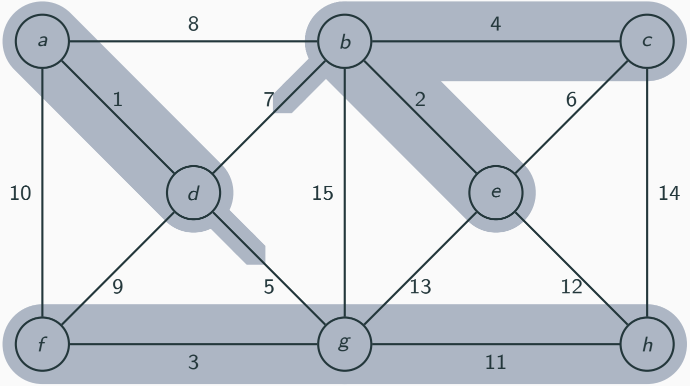

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

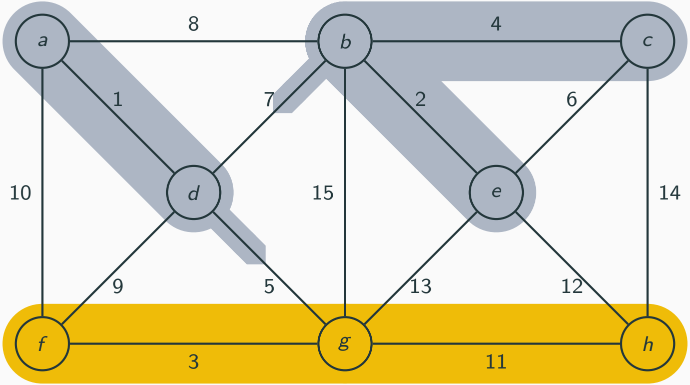

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

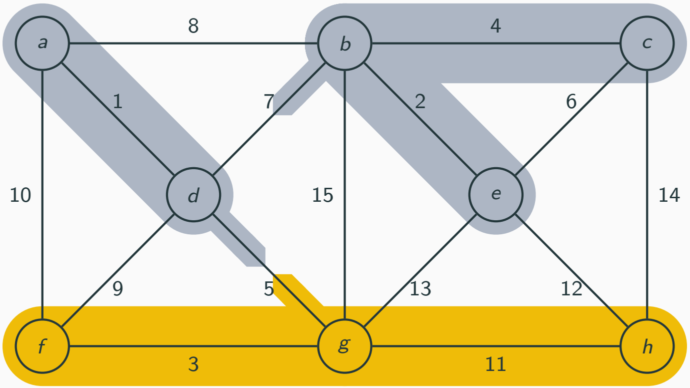

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

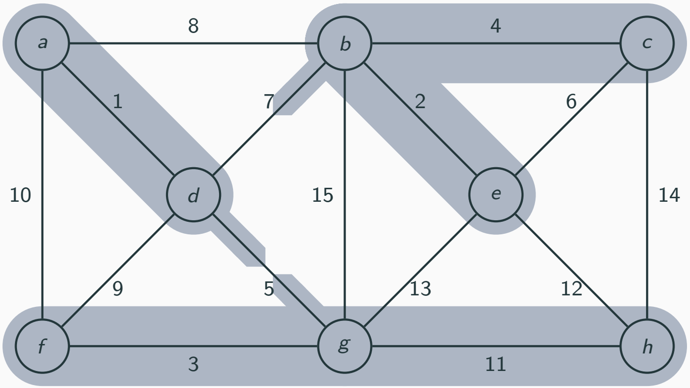

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

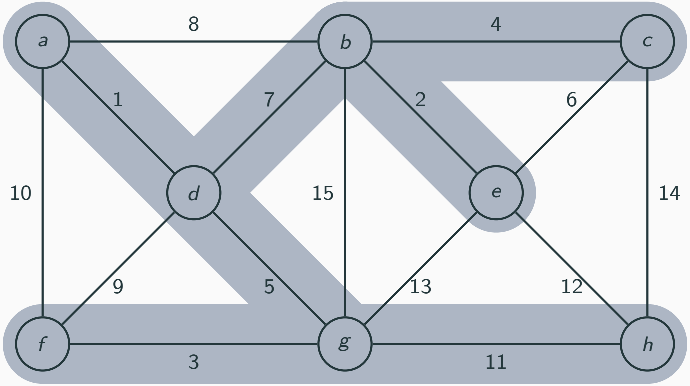

a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11


a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11


a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11


a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11


a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11


a
b
c
d
e
f
g
h
8
4
1
7
2
6
10
15
14
9
5
13
12
3
11

1 Algorithm

2 *m*-Kleene Algebras

3 Component Selection

4 Formalisation

5 Proof

6 Forests Modulo Equivalence

$$(\quad S \quad \sqcup \quad \sqcap \quad \bot \quad \top \quad \cdot \quad ^{-} \quad ^{\top} \quad 1 \quad * \quad + \quad s \quad m \quad)$$

Lattice

$$(S \; \sqcup \; \sqcap \; \bot \; \top \; \cdot \; ^{-} \; ^{\top} \; 1 \; ^{*} \; + \; s \; m)$$

Lattice

$$(\, S \;\; \sqcup \;\; \sqcap \;\; \bot \;\; \top \;\; \cdot \;\; {}^{-} \;\; {}^{\top} \;\; 1 \;\; {}^{*} \;\; + \;\; s \;\; m \,)$$

Bounded distributive lattice

Lattice

Stone relation algebra

$$(S \;\; \sqcup \;\; \sqcap \;\; \bot \;\; \top \;\; \cdot \;\; ^{-} \;\; ^{\top} \;\; 1 \;\; ^{*} \;\; + \;\; s \;\; m \;)$$

Bounded distributive lattice

Lattice

Stone relation algebra

$$(S \sqcup \sqcap \bot \top \cdot {}^{-} {}^{\top} 1 {}^{*} {}^{+} s\ m)$$

Bounded distributive lattice

Kleene algebra

Lattice

Stone relation algebra

$$(S \sqcup \sqcap \bot \top \cdot {}^{-} {}^{\top} 1 {}^{*} + s\ m)$$

Bounded distributive lattice

Kleene algebra

m-Kleene algebra

$$(\quad S \quad \sqcup \quad \sqcap \quad \bot \quad \top \quad \cdot \quad \overline{} \quad {}^{\top} \quad 1 \quad {}^{*} \quad + \quad s \quad m \quad)$$

$$\overline{\begin{bmatrix} \bot & 3 & \bot \\ \bot & \bot & 2 \\ 4 & \bot & \bot \end{bmatrix}} = \begin{bmatrix} \top & \bot & \top \\ \top & \top & \bot \\ \bot & \top & \top \end{bmatrix}$$

$$(\quad S \quad \sqcup \quad \sqcap \quad \bot \quad \top \quad \cdot \quad \overline{} \quad {}^{\top} \quad 1 \quad {}^{*} \quad + \quad s \quad m \quad)$$

$$\overline{\overline{\begin{bmatrix} \bot & 3 & \bot \\ \bot & \bot & 2 \\ 4 & \bot & \bot \end{bmatrix}}} = \begin{bmatrix} \bot & \top & \bot \\ \bot & \bot & \top \\ \top & \bot & \bot \end{bmatrix}$$

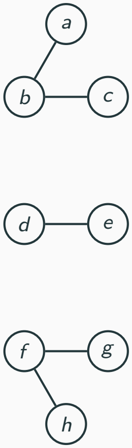
$$(\; S \; \sqcup \; \sqcap \; \bot \; \top \; \cdot \; {}^{-} \; {}^{\top} \; 1 \; {}^{*} \; + \; s \; m \;)$$

$$s \begin{bmatrix} \bot & 3 & \bot \\ \bot & \bot & 2 \\ 4 & \bot & \bot \end{bmatrix} = \begin{bmatrix} 9 & \bot & \bot \\ \bot & \bot & \bot \\ \bot & \bot & \bot \end{bmatrix}$$

$$(\; S \; \sqcup \; \sqcap \; \bot \; \top \; \cdot \; ^{-} \; ^{\top} \; 1 \; ^{*} \; + \; s \; m \;)$$

$$m\begin{bmatrix} \bot & 3 & \bot \\ \bot & \bot & 2 \\ 4 & \bot & \bot \end{bmatrix} = \begin{bmatrix} \bot & \bot & \bot \\ \bot & \bot & \top \\ \bot & \bot & \bot \end{bmatrix}$$

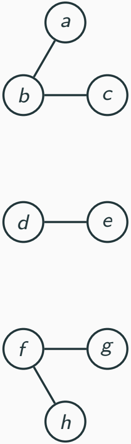
$k(x, v)$



$$k\ (x, v)$$

	a	b	c	d	e	f	g	h
a	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
b	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
c	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
d	⊥	⊥	⊥	⊤	⊤	⊥	⊥	⊥
e	⊥	⊥	⊥	⊤	⊤	⊥	⊥	⊥
f	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤
g	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤
h	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤

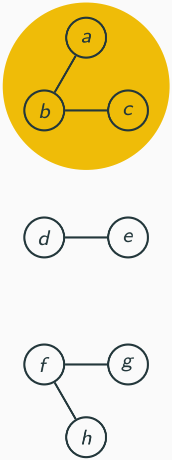
x



$k(x, v)$

	a	b	c	d	e	f	g	h
a	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
d	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
e	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
f	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\top$
g	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\top$
h	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\top$

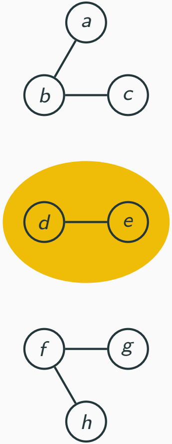
x



$$k(x, v)$$

	a	b	c	d	e	f	g	h
a	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
b	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
c	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
d	⊥	⊥	⊥	⊤	⊤	⊥	⊥	⊥
e	⊥	⊥	⊥	⊤	⊤	⊥	⊥	⊥
f	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤
g	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤
h	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤

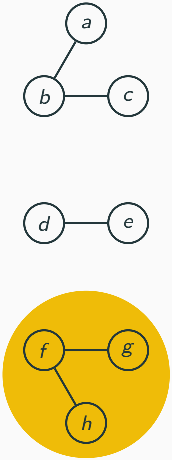
x



$$k(x, v)$$

	a	b	c	d	e	f	g	h
a	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
b	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
c	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥
d	⊥	⊥	⊥	⊤	⊤	⊥	⊥	⊥
e	⊥	⊥	⊥	⊤	⊤	⊥	⊥	⊥
f	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤
g	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤
h	⊥	⊥	⊥	⊥	⊥	⊤	⊤	⊤

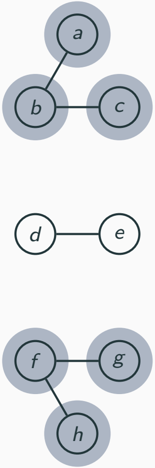
x



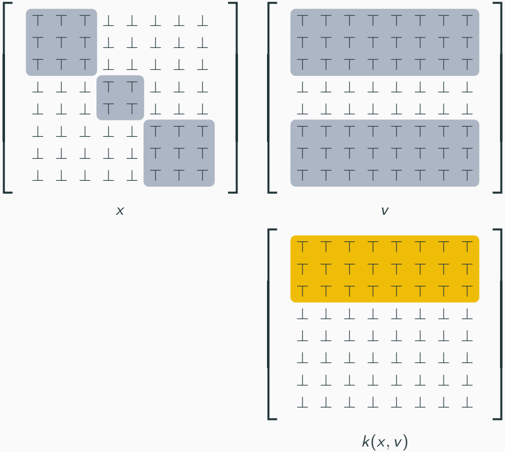
$$k(x, v)$$

	a	b	c	d	e	f	g	h
a	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
b	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
c	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
d	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
e	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
f	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
g	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
h	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤

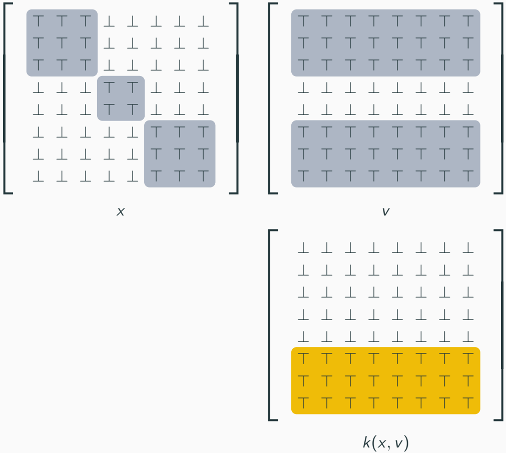
v



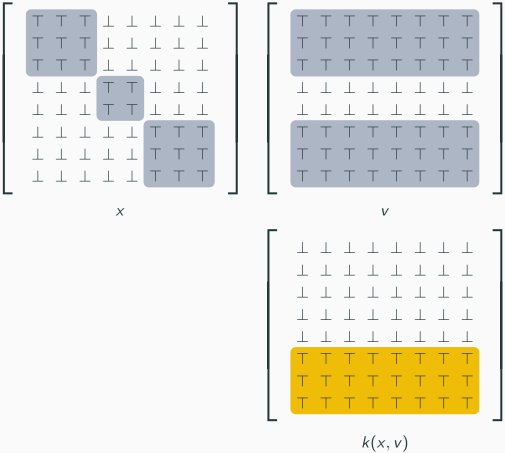
$k(x, v)$



$$k(x, v)$$

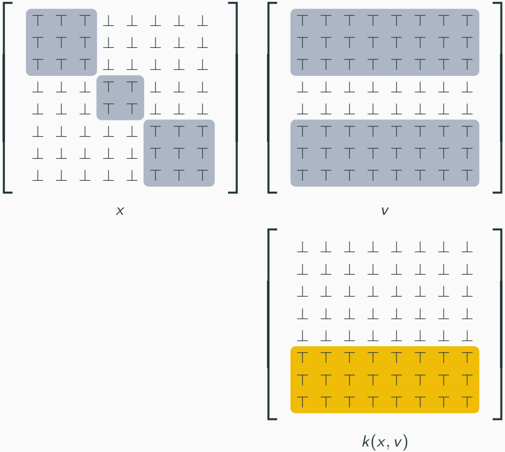


$k(x, v)$



$$k(x, v) = \overline{\overline{k(x, v)}}$$

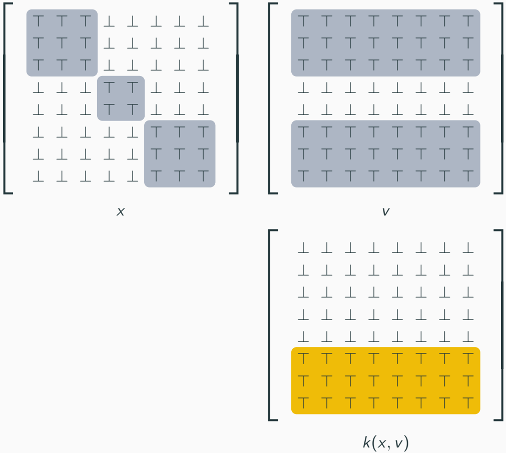
$k(x, v)$



$$k(x, v) = \overline{\overline{k(x, v)}}$$

$$k(x, v) = k(x, v) \cdot \top$$

$k(x, v)$



$$k(x, v) = \overline{\overline{k(x, v)}}$$

$$k(x, v) = k(x, v) \cdot \top$$

$$k(x, v) \leq \overline{\overline{v}}$$

$$k(\;x\;,\;v\;)$$

$$x = \begin{bmatrix} \top & \top & \top & \bot & \bot & \bot & \bot & \bot \\ \top & \top & \top & \bot & \bot & \bot & \bot & \bot \\ \top & \top & \top & \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \top & \top & \bot & \bot & \bot \\ \bot & \bot & \bot & \top & \top & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot & \top & \top & \top \\ \bot & \bot & \bot & \bot & \bot & \top & \top & \top \\ \bot & \bot & \bot & \bot & \bot & \top & \top & \top \end{bmatrix} \qquad v = \begin{bmatrix} \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top \\ \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\ \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top \end{bmatrix}$$

$$k(x, v) = \begin{bmatrix} \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\ \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top \end{bmatrix}$$

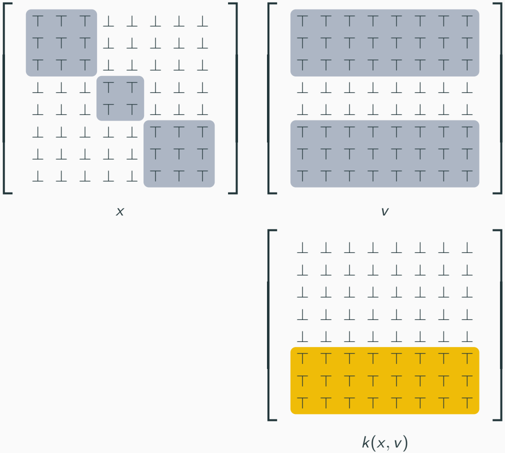
$$k(x, v) = \overline{\overline{k(x, v)}}$$

$$k(x, v) = k(x, v) \cdot \top$$

$$k(x, v) \leq \overline{\overline{v}}$$

$$k(x, v) \cdot k(x, v)^{\top} \leq x$$

$k(x, v)$



$$k(x,v) = \overline{\overline{k(x,v)}}$$

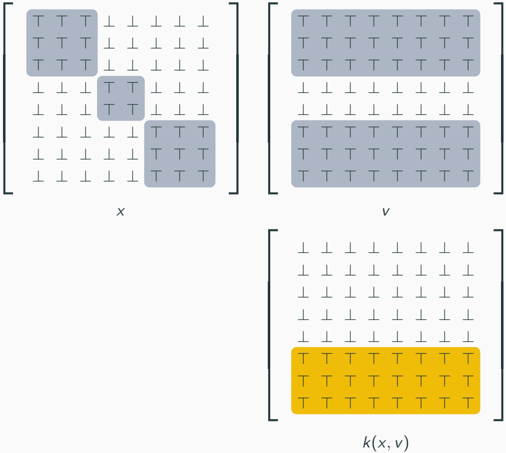
$$k(x,v) = k(x,v) \cdot \top$$

$$k(x,v) \leq \overline{\overline{v}}$$

$$k(x,v) \cdot k(x,v)^{\top} \leq x$$

$$x \cdot k(x,v) \leq k(x,v)$$

$$k(x, v)$$



$$k(x, v) = \overline{\overline{k(x, v)}}$$

$$k(x, v) = k(x, v) \cdot \top$$

$$k(x, v) \leq \overline{\overline{v}}$$

$$k(x, v) \cdot k(x, v)^{\top} \leq x$$

$$x \cdot k(x, v) \leq k(x, v)$$

$$k(x, v) \neq \bot$$

1. Algorithm
2. *m*-Kleene Algebras
3. Component Selection
4. Formalisation
5. Proof
6. Forests Modulo Equivalence

input g

$f \leftarrow \bot$

while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**

  $d \leftarrow \bot,\; h \leftarrow f,\; j \leftarrow \top$

  while $j \neq \bot$ **do**

    $v \leftarrow k(c(h),\, j)$

    $e \leftarrow m(v\overline{v}^{\top} \sqcap g)$

    if $e \leq \overline{f^{\top *} f^{*}}$ **then**

      $f \leftarrow f \sqcap \overline{e}^{\top}$

      $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$

      $d \leftarrow d \sqcup e$

    end

    $j \leftarrow j \sqcap \overline{v}$

  end

end

output f

input g

$f \leftarrow \bot$

while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**

  $d \leftarrow \bot,\ h \leftarrow f,\ j \leftarrow \top$

  while $j \neq \bot$ **do**

    $v \leftarrow k(c(h),\ j)$

    $e \leftarrow m(v\overline{v}^{\top} \sqcap g)$

    if $e \leq \overline{f^{\top *} f^{*}}$ **then**

      $f \leftarrow f \sqcap \overline{e}^{\top}$

      $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$

      $d \leftarrow d \sqcup e$

    end

    $j \leftarrow j \sqcap \overline{v}$

  end

end

output f

input g

$f \leftarrow \bot$

while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**

    $d \leftarrow \bot,\ h \leftarrow f,\ j \leftarrow \top$

    while $j \neq \bot$ **do**

        $v \leftarrow k(c(h),\ j)$

        $e \leftarrow m(v\overline{v}^{\top} \sqcap g)$

        if $e \leq \overline{f^{\top *} f^{*}}$ **then**

            $f \leftarrow f \sqcap \overline{e}^{\top}$

            $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$

            $d \leftarrow d \sqcup e$

        end

        $j \leftarrow j \sqcap \overline{v}$

    end

end

output f

input g
$f \leftarrow \bot$
while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**
  $d \leftarrow \bot,\; h \leftarrow f,\; j \leftarrow \top$
  while $j \neq \bot$ **do**
    $v \leftarrow k(c(h),\; j)$
    $e \leftarrow m(v\overline{v}^{\top} \sqcap g)$
    if $e \leq \overline{f^{\top *} f^{*}}$ **then**
      $f \leftarrow f \sqcap \overline{e}^{\top}$
      $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$
      $d \leftarrow d \sqcup e$
    end
    $j \leftarrow j \sqcap \overline{v}$
  end
end
output f

input g
$f \leftarrow \bot$
while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**
$d \leftarrow \bot,\ h \leftarrow f,\ j \leftarrow \top$
while $j \neq \bot$ **do**
$v \leftarrow k(c(h),\ j)$
$e \leftarrow m(v\overline{v}^{\top} \sqcap g)$
if $e \leq \overline{f^{\top *} f^{*}}$ **then**
$f \leftarrow f \sqcap \overline{e}^{\top}$
$f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$
$d \leftarrow d \sqcup e$
end
$j \leftarrow j \sqcap \overline{v}$
end
end
output f

input g

$f \leftarrow \bot$

while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**

  $d \leftarrow \bot,\; h \leftarrow f,\; j \leftarrow \top$

  while $j \neq \bot$ **do**

    $v \leftarrow k(c(h),\; j)$

    $e \leftarrow m(v\overline{v}^{\top} \sqcap g)$

    if $e \leq \overline{f^{\top *} f^{*}}$ **then**

      $f \leftarrow f \sqcap \overline{e}^{\top}$

      $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$

      $d \leftarrow d \sqcup e$

    end

    $j \leftarrow j \sqcap \overline{v}$

  end

end

output f

input g
$f \leftarrow \bot$
while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**
  $d \leftarrow \bot,\ h \leftarrow f,\ j \leftarrow \top$
  while $j \neq \bot$ **do**
    $v \leftarrow k(c(h),\ j)$
    $e \leftarrow m(v\overline{v}^{\top} \sqcap g)$
    if $e \leq \overline{f^{\top *} f^{*}}$ **then**
      $f \leftarrow f \sqcap \overline{e}^{\top}$
      $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$
      $d \leftarrow d \sqcup e$
    end
    $j \leftarrow j \sqcap \overline{v}$
  end
end
output f

input g

$f \leftarrow \bot$

while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**

  $d \leftarrow \bot,\ h \leftarrow f,\ j \leftarrow \top$

  while $j \neq \bot$ **do**

    $v \leftarrow k(c(h),\ j)$

    $e \leftarrow m(v \overline{v}^{\top} \sqcap g)$

    if $e \leq \overline{f^{\top *} f^{*}}$ **then**

      $f \leftarrow f \sqcap \overline{e}^{\top}$

      $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$

      $d \leftarrow d \sqcup e$

    end

    $j \leftarrow j \sqcap \overline{v}$

  end

end

output f

input g

$f \leftarrow \bot$

while $\overline{f^{\top *} f^{*}} \sqcap g \neq \bot$ **do**

    $d \leftarrow \bot, \; h \leftarrow f, \; j \leftarrow \top$

    while $j \neq \bot$ **do**

        $v \leftarrow k(c(h), \; j)$

        $e \leftarrow m(v\overline{v}^{\top} \sqcap g)$

        if $e \leq \overline{f^{\top *} f^{*}}$ **then**

            $f \leftarrow f \sqcap \overline{e}^{\top}$

            $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$

            $d \leftarrow d \sqcup e$

        end

        $j \leftarrow j \sqcap \overline{v}$

    end

end

output f

```
input $g$
$f \leftarrow \bot$
while $\overline{f^{\top *} f^*} \sqcap g \neq \bot$ do
    $d \leftarrow \bot,\ h \leftarrow f,\ j \leftarrow \top$
    while $j \neq \bot$ do
        $v \leftarrow k(c(h),\ j)$
        $e \leftarrow m(v\overline{v}^{\top} \sqcap g)$
        if $e \leq \overline{f^{\top *} f^*}$ then
            $f \leftarrow f \sqcap \overline{e}^{\top}$
            $f \leftarrow (f \sqcap \overline{\top e f^{\top *}}) \sqcup (f \sqcap \top e f^{\top *})^{\top} \sqcup e$
            $d \leftarrow d \sqcup e$
        end
        $j \leftarrow j \sqcap \overline{v}$
    end
end
output $f$
```

1 Algorithm

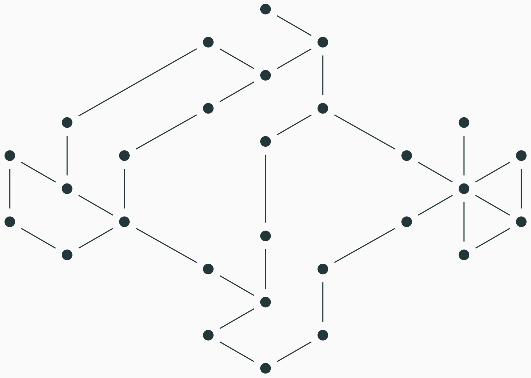
2 *m*-Kleene Algebras

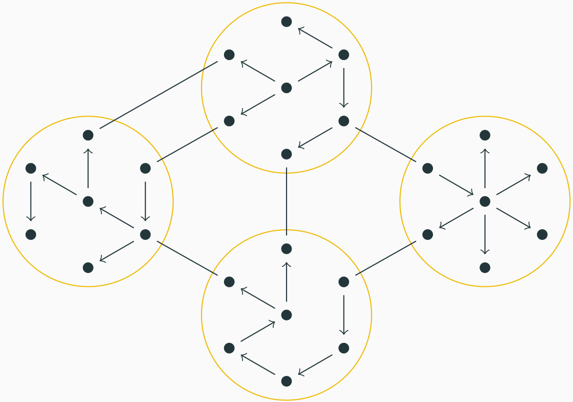
3 Component Selection

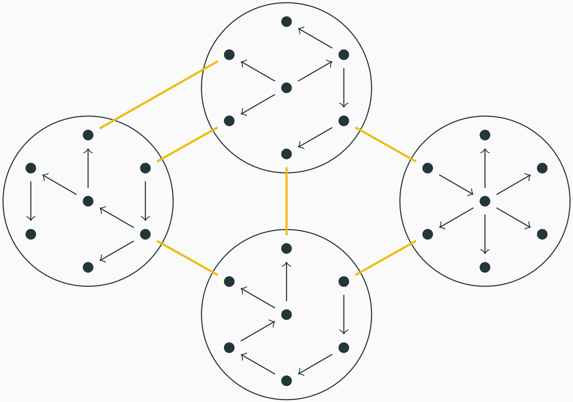
4 Formalisation

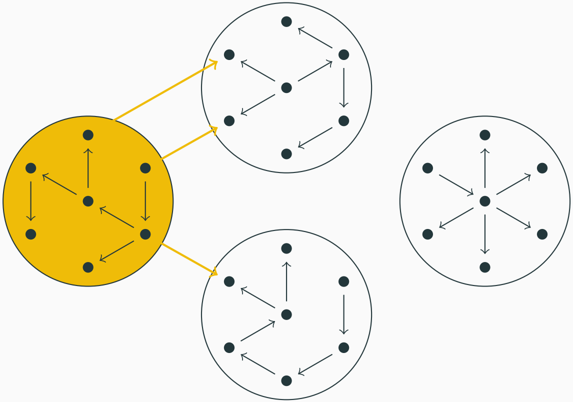
5 Proof

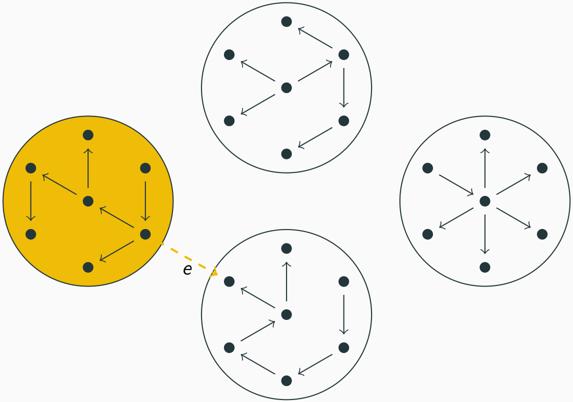
6 Forests Modulo Equivalence

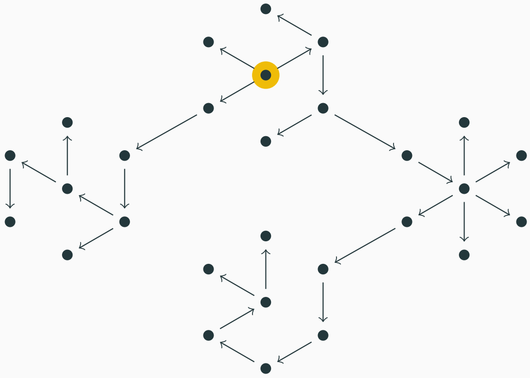


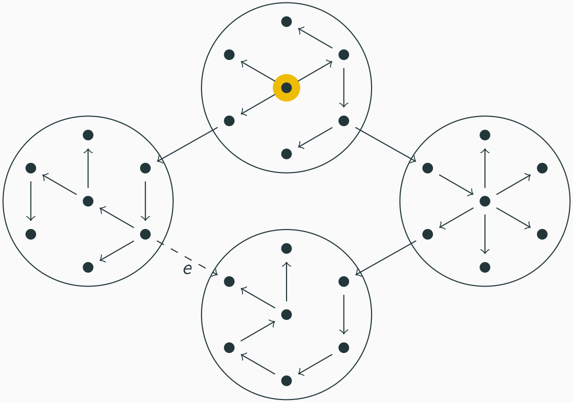


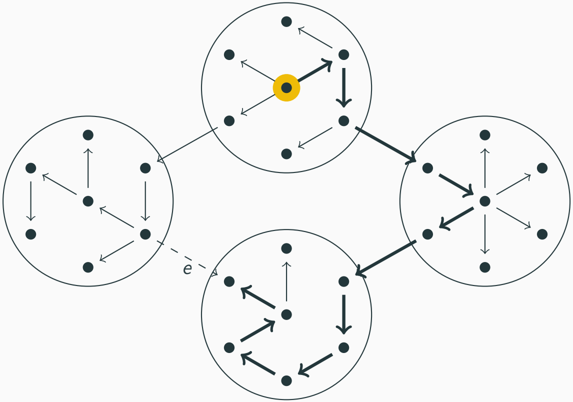


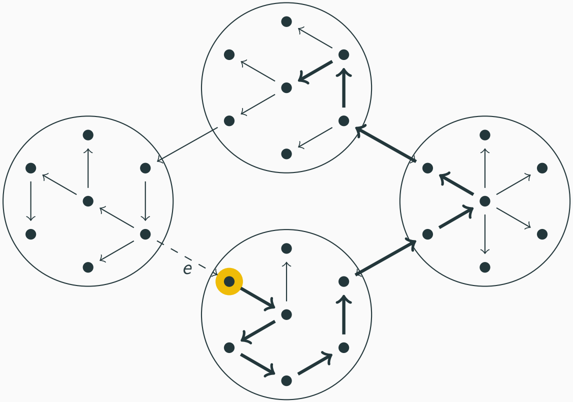


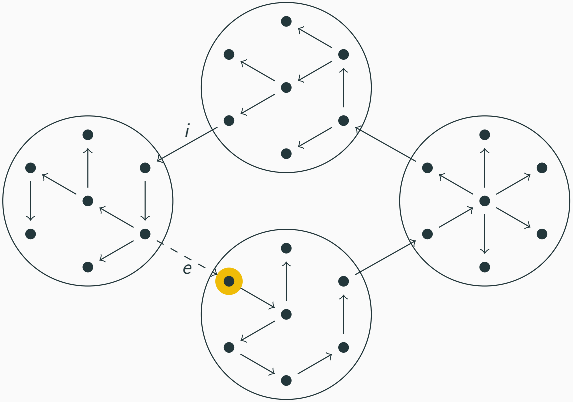

e

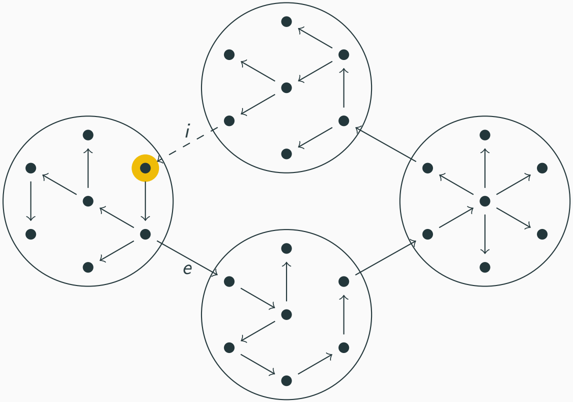


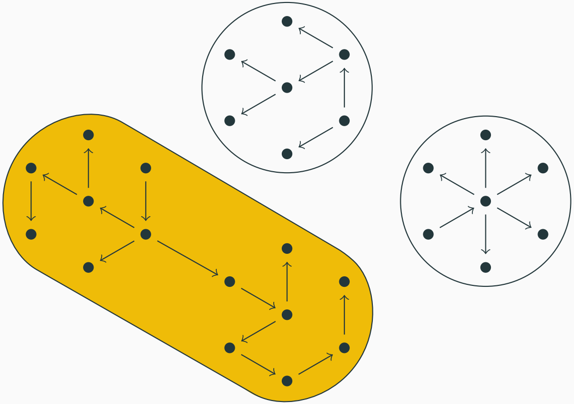

e


e

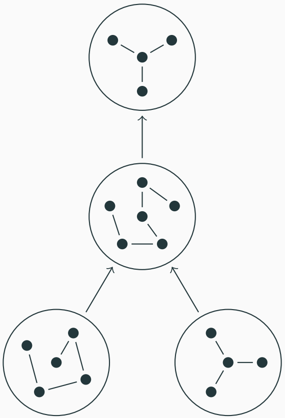

e


i
e

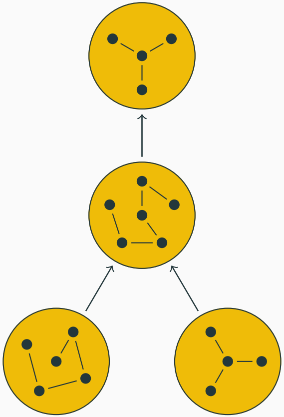

i
e



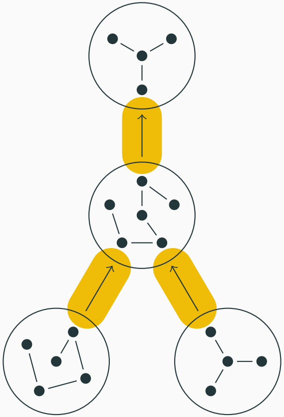
d forest modulo x



d forest modulo x

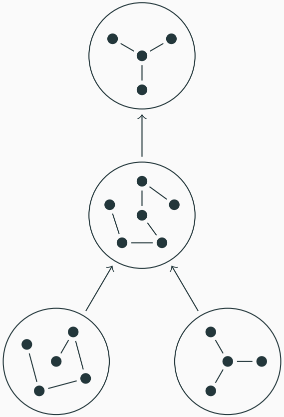


d forest modulo x



d forest modulo x

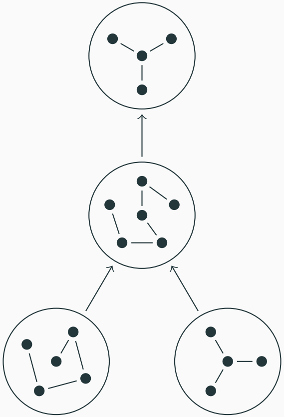
$$(xd)^\top xd \leq 1$$



d forest modulo x

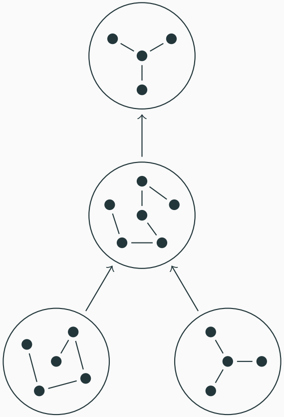
$$(xd)^\top xd \leq 1$$

$$x \sqcap dd^\top \leq 1$$

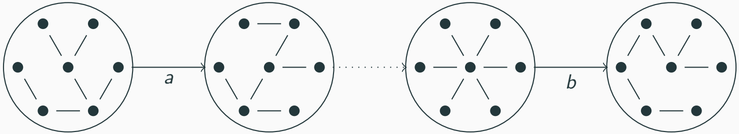


d forest modulo x

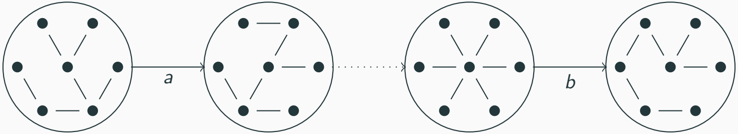
$$(xd)^\top xd \leq 1$$
$$x \sqcap dd^\top \leq 1$$
$$x \sqcap (xd)^+ \leq \bot$$



d forest modulo x

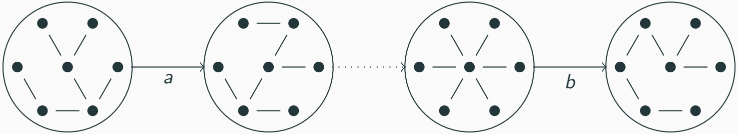


d forest modulo x



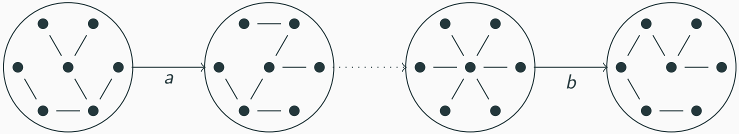
$$\forall a, b: \; a \rightsquigarrow^{d}_{c(h)} b \;\wedge\; a \leq \overline{c(h)} \sqcap \overline{\overline{g}} \;\wedge\; b \leq d \;\Rightarrow\; s(b \sqcap g) \leq s(a \sqcap g)$$

d forest modulo x



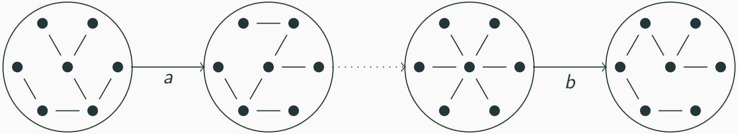
$$\forall a, b : a \rightsquigarrow^{d}_{c(h)} b \wedge a \leq \overline{c(h)} \sqcap \overline{\overline{g}} \wedge b \leq d \Rightarrow s(b \sqcap g) \leq s(a \sqcap g)$$

d forest modulo x



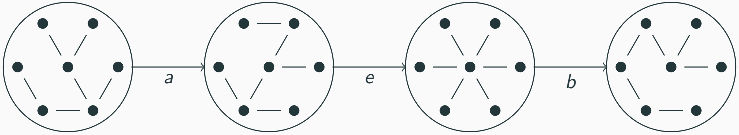
$$\forall a, b : \; a \rightsquigarrow^{d}_{c(h)} b \;\wedge\; a \leq \overline{c(h)} \sqcap \overline{\overline{g}} \;\wedge\; b \leq d \Rightarrow s(b \sqcap g) \leq s(a \sqcap g)$$

d forest modulo x



$$\forall a, b : \; a \rightsquigarrow^{d}_{c(h)} b \;\wedge\; a \leq \overline{c(h)} \sqcap \overline{\overline{g}} \;\wedge\; b \leq d \;\Rightarrow\; s(b \sqcap g) \leq s(a \sqcap g)$$

d forest modulo x



$$a \rightsquigarrow_x^{d \sqcup e} b \Leftrightarrow a \rightsquigarrow_x^{d} b \vee \left(a \rightsquigarrow_x^{d} e \wedge e \rightsquigarrow_x^{d} b \right)$$

Key Abstractions

- Component selection $k(x, v)$
- Forest modulo equivalence
- $a \rightsquigarrow^d_x b$

Key Abstractions

- Component selection $k(x, v)$
- Forest modulo equivalence
- $a \rightsquigarrow^{d}_{x} b$

Future Work

- Alter formalisation and proof to be undirected
- Use algebraic framework for other graph related algorithms