Experimental Investigation of Sufficient Criteria for Relations to Have Kernels

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Introduction

When written as a logical formula, most mathematical theorems have the following form:

$$\boxed{\forall x : \Phi(x) \Rightarrow \Psi(x)}$$

In such a case:

- x is a list of variables.
- Each variable ranges over a certain class of mathematical objects.
- $\Psi(x)$ describes the property one is actually interested in.
- $\Phi(x)$ describes a property that ensures $\Psi(x)$.

Mostly, one tries to get $\Phi(x)$ as general as possible.

In lucky cases $\Phi(x)$ can be shown to be equivalent to $\Psi(x)$. Then it even **characterises** the class of mathematical objects for which $\Psi(x)$ holds, i.e., those one is interested in.

Example: Fixpoint theorem of A. Tarski.

- Only one variable x that ranges over the class of lattices.
- $\Psi(x)$ describes that each monotonic function on x has a least fixpoint.
- $\Phi(x)$ describes that x is complete.

That $\Phi(x)$ and $\Psi(x)$ are equivalent follows from a theorem of A. Davis.

Further examples: Characterisations of classes of mathematical objects by means of forbidden substructures.

- A lattice is **modular** iff it does not contain a sublattice isomorphic to the pentagon-lattice N_5 (R. Dedekind).
- A finite graph is **planar** iff it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$ (K. Kuratowski).
- ...

Our Problem

A **kernel** of a directed graph $G = (X, R)$ is a subset K of the set X of vertices with the following two properties:

- No pair of vertices of K is connected by an edge (K **stable**).
- Each vertex not in K is connected to one of K (K **absorbant**).

V. Chvatal has shown that determining whether a directed graph possesses a kernel is NP-complete.

Mapping formula $\forall x : \Phi(x) \Rightarrow \Psi(x)$ to the existence problem of kernels:

- x ranges over the class of directed graphs.
- $\Psi(x)$ describes that x has a kernel.
- $\Phi(x)$ describes a sufficient criterion for x to have a kernel.

Due to Chvatal's result it is very unlikely to get a $\Phi(x)$ such that $\Phi(x)$ and $\Psi(x)$ are equivalent and $\Phi(x)$ can be computed efficiently.

There exist a series of sufficient criteria for the existence of kernels which can be tested efficiently.

We want to answer the question **how close these are to a characterisation of the class of directed graphs having kernels**.

We investigate the most prominent criteria:

- The relation of the graph is **irreflexive** and **symmetric**.
- The graph is **progressively finite** (informal: there is no infinite path $(x_1, x_2, x_3, \ldots)$; in the finite case this means **to be cyclefree**).
- The graph is **bipartite**.
- The graph has **no cycles of odd length** (Richardson's criterion).

The first and last criterion requires the graph to be finite, the second and third criterion also holds for infinite graphs.

The Experimental Results

Occurrences of kernels within directed graphs having at most 7 vertices.

$\|X\|$	# all relations	# relations with kernel	percentage
1	2	1	50.00 %
2	16	8	50.00 %
3	512	230	44.92 %
4	65 536	26 346	40.19 %
5	33 554 432	12 378 964	39.98 %
6	68 719 476 736	23 921 882 920	34.80 %
7	562 949 953 421 312	188 553 949 010 868	33.49 %

Conjecture: If k_n denotes the number of directed graphs with $1, \ldots, n$ as vertices that have a kernel, then:

$$\lim_{n \to \infty} \frac{k_n}{2^{n^2}} = 0$$

Number of directed graphs for the four criteria having at most 7 vertices.

$\|X\|$	# irr., symm.	# progr. finite	# bipartite	# Richardson
1	1	1	1	1
2	2	3	4	4
3	8	25	37	49
4	64	543	829	1 699
5	1 024	29 281	36 616	150 736
6	32 768	3 781 503	3 327 499	32 398 249
7	2 097 152	1 138 779 265	581 809 537	16 230 843 049

The numerical data show that even the most general of the four criteria, Richardson's criterion, **is very far away from a characterisation** of the class of directed graphs having kernels.

E.g., in case of 7 vertices Richardson's criterion holds only for 0.008 % of the directed graphs having kernels.

The Approach

To get the numbers of the tables for a set X, we proceeded as follows:

1. We developed relational specifications of vectors of type

$$[X \leftrightarrow X] \leftrightarrow \mathbf{1}$$

   (with $[X \leftrightarrow X]$ as set of relations on X) that model
   - the set of relations on X which have a kernel,
   - the set of relations on X which are irreflexive and symmetric,
   - the set of relations on X which are bipartite,
   - the set of relations on X which are progressively finite,
   - the set of relations on X which have no cycles of odd length.
2. We translated the relational specifications into RelView-programs
3. We executed the programs by RelView.

Because of the very efficient ROBDD-implementation of relations by Rel View in all cases an execution was possible up to $|X| = 7$. The numbers are part of the representation of the results.

The Relational Constructions we Need

- **Basic operations** and **basic constants** of relation algebra.

$$R \cup S \qquad R \cap S \qquad R;S \qquad \overline{R}, \qquad R^{\mathsf{T}} \qquad \mathsf{O} \qquad \mathsf{L} \qquad \mathsf{I}$$

- **Right residual** $R \setminus S := \overline{R^{\mathsf{T}};\overline{S}}$ (drived operation). Pointwise description:

$$(R \setminus S)_{y,z} \iff \forall x : R_{x,y} \Rightarrow S_{x,z}$$

- **Projection relations** $\pi : X \times Y \leftrightarrow X$ and $\rho : X \times Y \leftrightarrow Y$ of $X \times Y$ as further basic constants. Pointwise description:

$$\pi_{u,x} \iff u_1 = x \qquad \rho_{u,y} \iff u_2 = y$$

- **Left pairing** $[\![R, S]\!] := \pi;R \cap \rho;S$, **right pairing** $[R, S]\!] := [\![R^{\mathsf{T}}, S^{\mathsf{T}}]^{\mathsf{T}}$ and **parallel composition** $R \,\|\, S := [\pi;R, \rho;S]\!]$ (derived operations). Pointwise descriptions:

$$[\![R, S]_{u,z} \iff R_{u_1,z} \wedge S_{u_2,z}$$
$$[R, S]\!]_{z,u} \iff R_{z,u_1} \wedge S_{z,u_2}$$
$$(R \,\|\, S)_{u,v} \iff R_{u_1,v_1} \wedge S_{u_2,v_2}$$

- **Membership relations** $\mathsf{M} : X \leftrightarrow 2^X$ as further basic constants. Pointwise description:

$$\mathsf{M}_{x,Y} \iff x \in Y$$

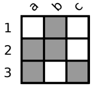
If X is a direct product $Y \times Z$, we use $\mathbf{M}$ instead of M and get:

$$\mathbf{M}_{u,R} \iff R_{u_1,u_2}$$

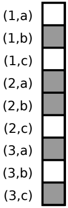
- **Vector model** $\mathrm{vec}(R) = [\![R, \mathsf{I}]\!];\mathsf{L}$ (derived operation). Pointwise description:

$$\mathrm{vec}(R)_u \iff R_{u_1,u_2}$$

Example (produced by RelView):



$R : X \leftrightarrow Y$



$\mathrm{vec}(R) : X \times Y \leftrightarrow \mathbf{1}\ (= \{\bot\})$

Example: Relations Having Kernels

To get a relational specification of a vector

$$\mathfrak{kernel}(X) : [X \leftrightarrow X] \leftrightarrow \mathbf{1}$$

that models the set of relations R on the set X such that the directed graph $G = (X, R)$ has a kernel, we start as follows (A ranges over 2^X):

$$\begin{aligned}
\mathfrak{kernel}(X)_R &\iff \exists\, A : \mathfrak{stable}(X)_{A,R} \wedge \mathfrak{absorb}(X)_{A,R} \\
&\iff \exists\, A : \mathsf{L}_{\perp,A} \wedge (\mathfrak{stable}(X) \cap \mathfrak{absorb}(X))_{A,R} \\
&\iff (\mathsf{L};(\mathfrak{stable}(X) \cap \mathfrak{absorb}(X)))_{\perp,R}
\end{aligned}$$

Hence, we have

$$\boxed{\mathfrak{kernel}(X) := (\mathsf{L};(\mathfrak{stable}(X) \cap \mathfrak{absorb}(X)))^{\mathsf{T}}}$$

and it remains the development of the auxiliary specifications

$$\mathfrak{stable}(X) : 2^X \leftrightarrow [X{\leftrightarrow}X] \qquad \mathfrak{absorb}(X) : 2^X \leftrightarrow [X{\leftrightarrow}X].$$

For the development of the auxiliary specifications we assume arbitrary $A \in 2^X$ and $R : X \leftrightarrow X$. In the first case we get (u ranges over X^2):

$$\begin{aligned}
\mathfrak{stable}(X)_{A,R} &\iff A \text{ is stable in } G = (X,R) \\
&\iff \neg\exists\, u : u_1 \in A \wedge u_2 \in A \wedge R_{u_1,u_2} \\
&\iff \neg\exists\, u : \mathsf{M}_{u_1,A} \wedge \mathsf{M}_{u_2,A} \wedge R_{u_1,u_2} \\
&\iff \neg\exists\, u : \mathsf{M}^{\mathsf{T}}_{A,u_1} \wedge \mathsf{M}^{\mathsf{T}}_{A,u_2} \wedge R_{u_1,u_2} \\
&\iff \neg\exists\, u : [\mathsf{M}^{\mathsf{T}}, \mathsf{M}^{\mathsf{T}}]\!]_{A,u} \wedge \mathbf{M}_{u,R} \\
&\iff \overline{[\mathsf{M}^{\mathsf{T}}, \mathsf{M}^{\mathsf{T}}]\!];\mathbf{M}}_{A,R}
\end{aligned}$$

Here $\mathsf{M} : X \leftrightarrow 2^X$ and $\mathbf{M} : X^2 \leftrightarrow [X \leftrightarrow X]$ are two membership relations. Thr calculation yields:

$$\boxed{\mathfrak{stable}(X) := \overline{[\mathsf{M}^{\mathsf{T}}, \mathsf{M}^{\mathsf{T}}]\!];\mathbf{M}}}$$

In the second case we get (x, y range over X, and u ranges over X^2 and B ranges over 2^X):

$$
\begin{aligned}
&\mathfrak{absorb}(X)_{A,R}\\
\iff\ & A \text{ is absorbant in } G = (X, R)\\
\iff\ & \forall x : x \notin A \Rightarrow \exists\, y : y \in A \wedge R_{x,y}\\
\iff\ & \forall x, B : B = A \wedge x \notin B \Rightarrow \exists\, u : x = u_1 \wedge u_2 \in B \wedge R_{u_1,u_2}\\
\iff\ & \forall x, B : B = A \wedge \overline{\mathsf{M}}_{x,B} \Rightarrow \exists\, u : \mathsf{I}_{x,u_1} \wedge \mathsf{M}^{\mathsf{T}}_{B,u_2} \wedge R_{u_1,u_2}\\
\iff\ & \forall x, B : B = A \wedge \overline{\mathsf{M}}_{x,B} \Rightarrow \exists\, u : (\mathsf{I} \,\|\, \mathsf{M}^{\mathsf{T}})_{(x,B),u} \wedge \mathbf{M}_{u,R}\\
\iff\ & \forall x, B : \beta_{(x,B),A} \wedge \mathrm{vec}(\overline{\mathsf{M}})_{(x,B)} \Rightarrow ((\mathsf{I} \,\|\, \mathsf{M}^{\mathsf{T}});\mathbf{M})_{(x,B),R}\\
\iff\ & \forall x, B : (\beta \cap \mathrm{vec}(\overline{\mathsf{M}});\mathsf{L})_{(x,B),A} \Rightarrow ((\mathsf{I} \,\|\, \mathsf{M}^{\mathsf{T}});\mathbf{M})_{(x,B),R}\\
\iff\ & ((\beta \cap \mathrm{vec}(\overline{\mathsf{M}});\mathsf{L}) \setminus ((\mathsf{I} \,\|\, \mathsf{M}^{\mathsf{T}});\mathbf{M}))_{A,R}
\end{aligned}
$$

Here β is the second projection relation of $X \times 2^X$. Altogether:

$$\mathfrak{absorb} := (\beta \cap \mathrm{vec}(\overline{\mathsf{M}});\mathsf{L}) \setminus ((\mathsf{I} \,\|\, \mathsf{M}^{\mathsf{T}});\mathbf{M})$$

The RelView-implementation of $\mathfrak{kernel}(X)$, where X is a homogeneous relation that provides the set X:

```
Kernel(X)
  DECL M, MM, beta, stable, absorb
  BEG  M = epsi(X);
       MM = epsi(pr1(X,X));
       beta = pr2(X,M^*M);
       stable = -([M^,M^|]*MM);
       absorb = (beta & vec(-M)*L1n(M)) \ par(I(X),M^)*MM
       RETURN (L1n(M)*(stable & absorb))^
  END.
```

Used auxiliary programs:

- pr1(X,Y) for the first projection of $X \times Y$.
- pr2(X,Y) for the second projection of $X \times Y$.
- par(R,S) for the parallel composition.
- vec(R) for vec.

Example: Relations Being Bipartite

Assume an arbitrary $R : X \leftrightarrow X$. Then we have (A ranges over 2^X and u ranges over X^2):

$$
\begin{aligned}
& \mathfrak{bipartite}(X)_R \\
\iff\ & G = (X, R) \text{ is bipartite} \\
\iff\ & \exists\, A : \forall u : R_{u_1,u_2} \Rightarrow (u_1 \in A \wedge u_2 \notin A) \vee (u_1 \notin A \wedge u_2 \in A) \\
\iff\ & \exists\, A : \forall u : R_{u_1,u_2} \Rightarrow (\mathsf{M}_{u_1,A} \wedge \overline{\mathsf{M}}_{u_2,A}) \vee (\overline{\mathsf{M}}_{u_1,A} \wedge \mathsf{M}_{u_2,A}) \\
\iff\ & \exists\, A : \forall u : \mathbf{M}_{u,R} \Rightarrow [\![\mathsf{M}, \overline{\mathsf{M}}]\!]_{u,A} \vee [\![\overline{\mathsf{M}}, \mathsf{M}]\!]_{u,A} \\
\iff\ & \exists\, A : \forall u : \mathbf{M}_{u,R} \Rightarrow ([\![\mathsf{M}, \overline{\mathsf{M}}]\!] \cup [\![\overline{\mathsf{M}}, \mathsf{M}]\!])_{u,A} \\
\iff\ & \exists\, A : (\mathbf{M} \setminus ([\![\mathsf{M}, \overline{\mathsf{M}}]\!] \cup [\![\overline{\mathsf{M}}, \mathsf{M}]\!]))_{R,A} \wedge \mathsf{L}_A \\
\iff\ & ((\mathbf{M} \setminus ([\![\mathsf{M}, \overline{\mathsf{M}}]\!] \cup [\![\overline{\mathsf{M}}, \mathsf{M}]\!]));\mathsf{L})_R
\end{aligned}
$$

Resulting relational specification $\mathfrak{bipartite}(X) : [X \leftrightarrow X] \leftrightarrow \mathbf{1}$:

$$\mathfrak{bipartite}(X) := (\mathbf{M} \setminus ([\![\mathsf{M}, \overline{\mathsf{M}}]\!] \cup [\![\overline{\mathsf{M}}, \mathsf{M}]\!]));\mathsf{L}$$

The RelView-implementation of $\mathfrak{bipartite}(X)$, where `X` is a homogeneous relation that provides the set X:

```
Bipartite(X)
  DECL M, MM
  BEG  M = epsi(X);
       MM = epsi(pr1(X,X))
       RETURN dom(MM \ ([|M,-M] | [|-M,M]))
  END.
```

The Running Times (in Seconds)

PC with Intel Xeon E5-2698 CPU, 3.60 GHz, 512 GByte, Arch Linux 5.2.0.

$\lvert X \rvert$	irr., sym.	prog. fin.	bipart.	Richardson	a kernel	all rel.
1	0.0010	0.0012	0.0009	0.0012	0.0015	0.0006
2	0.0026	0.0032	0.0018	0.0067	0.0057	0.0007
3	0.0069	0.0082	0.0053	0.0117	0.0117	0.0007
4	0.0081	0.0172	0.0194	0.0150	0.0171	0.0008
5	0.0169	0.0262	0.0199	0.1807	0.0213	0.0010
6	0.0181	0.1211	0.0833	10.4710	0.3141	0.0011
7	0.0476	1.8771	2.3501	32220.5500	138.6700	0.0011

Homepage of RelView:

`https://www.rpe.informatik.uni-kiel.de/en/research/relview`

Source code of RelView, version 8.2:

`https://github.com/relview`

Conclusion

There exist some extensions of Richardson's theorem which allow the existence of cycles of odd length, e.g.:

- All edges (x, y) of a cycle of odd length are **symmetric**, i.e., also (y, x) is an edge of the graph (C. Berge and P. Duchet).
- All cycles of odd length have at least two symmetric edges (P. Duchet).

Testing the criteria seems to be rather expensive since it requires to check all cycles of odd length.

To get a feeling for the behaviour, we have applied our approach to them.

- The criterion of Berge and Duchet is only slightly more general than Richardson's.
- Duchet's criterion is much more general than Richardson's. Nevertheless it is still far away from a characterisation of the directed graphs with kernels. E.g., for $|X| = 6$ it holds only for 0.9% of the graphs.