Some Modal and Temporal Translations of Generalized Basic Logic

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The Gödel-McKinsey-Tarski Translation

- The **Gödel-McKinsey-Tarski translation** is one of the fundamental ideas of intuitionistic logic and modal logic.
- It interprets intuionistic logic in the classical modal logic $S4$.
- Algebraic perspective is illuminating: The algebraic models of intuitionistic logic and $S4$ are respectively Heyting algebras and interior algebras (i.e., Boolean algebras with an interior operator).
- Algebraically, the GMT translation says that we can view Heyting algebras as the elements of interior algebras such that $\Box x = x$, where $\Box$ is the interior operator.

The Main Idea

- The content of our paper: The same set-up applies when intuitionistic logic replaced by **generalized basic logic** and classical logic is replaced by **Łukasiewicz logic**.
- Extends the classical translation result to an important substructural setting, contributing to the general program of extending intuitionsitic results to substructural logics.
- Links some prominent substructural logics, adding to our understanding of the general structure of substructural logics.
- This work is based on: F. 2021 'Poset Products as Relational Models' and the work of P. Jipsen and F. Montagna on the **poset product** construction.

History of **GBL**

- **Generalized basic logic** arose out of efforts among algebraic logicians to extend Hájek's basic fuzzy logic.
- Idea is to extend BL-algebras to encompass Heyting algebras, lattice-ordered groups, their negative cones, and other algebras in the vicinity.
- In the case with exchange, weakening, and falsum (lower bound), generalized basic logic is a natural common fragment of basic logic and intuitionistic logic.
- We first discuss **GBL** from a logical point of view, and fix a countable set Var of propositional variable symbols and a basic language $\mathcal{L} = \{\wedge, \vee, \cdot, \rightarrow, 0, 1\}$.

Hilbert Systems for **GBL**, **BL**, and **Ł**

(A1) $\varphi \to \varphi$

(A2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A3) $(\varphi \cdot \psi) \to (\psi \cdot \varphi)$

(A4) $(\varphi \cdot \psi) \to \psi$

(A5) $(\varphi \to (\psi \to \chi)) \to ((\varphi \cdot \psi) \to \chi))$

(A6) $((\varphi \cdot \psi) \to \chi)) \to (\varphi \to (\psi \to \chi))$

(A7) $(\varphi \cdot (\varphi \to \psi)) \to (\varphi \wedge \psi)$

(A8) $(\varphi \wedge \psi) \to (\varphi \cdot (\varphi \to \psi))$

(A9) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

(A10) $\varphi \to (\varphi \vee \psi)$

(A11) $\psi \to (\varphi \vee \psi)$

(A12) $((\varphi \to \psi) \wedge (\chi \to \psi)) \to ((\varphi \vee \chi) \to \psi)$

(A13) $0 \to \varphi$

(MP) $\varphi, \varphi \to \psi \vdash \psi$

(P) $(\varphi \to \psi) \vee (\psi \to \varphi)$

(DN) $\neg\neg\varphi \leftrightarrow \varphi$ (usual abbreviations apply).

Let I be a fresh set of unary connective symbols (intended as $\Box$-modals). We introduce a new family of logics Ł(I) by adding to our calculus for Ł the axioms

($K_\Box$) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

($P_\Box$) $\Box(\varphi \cdot \psi) \leftrightarrow \Box\varphi \cdot \Box\psi$

($M_\Box$) $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$

($\Box 1$) $\Box 1 \leftrightarrow 1$

($\Box 0$) $\Box 0 \leftrightarrow 0$

($\Box$-Nec) $\varphi \vdash \Box\varphi$

The logic $S4$Ł(I) is obtained by further adding:

($T_\Box$) $\Box\varphi \to \varphi$

($4_\Box$) $\Box\varphi \to \Box\Box\varphi$

If $I = \{G, H\}$, then we obtain the logic $S4_t$Ł by adding to $S4$Ł(G, H) the axioms:

(GP) $\varphi \to G\neg H\neg\varphi$

(HF) $\varphi \to H\neg G\neg\varphi$

Temporal Logic and Some Abbreviations

We define diamonds by $\Diamond = \neg\Box\neg$ as usual. In$S4_t$Ł, the diamond connectives P and F are abbreviations for $\neg H \neg$ and $\neg G \neg$, respectively. The typical intended interpretations of the modals G, P, H, F are:

- $G\varphi$: "It is always **g**oing to be the case that φ."
- $P\varphi$: "It was true at one point in the **p**ast that φ."
- $H\varphi$: "It always **h**as been the case that φ."
- $F\varphi$: "It will be true at some point in the **f**uture that φ."

(This descends from Prior's tense logic). We also just denote $S4$Ł$(\Box)$ by $S4$Ł.

Residuated Lattices

Definition:

A **bounded commutative integral residuated lattice** is an algebra $(A, \wedge, \vee, \cdot, \to, 0, 1)$ such that

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.
- $(A, \cdot, 1)$ is a commutative monoid.
- For all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq y \to z.$$

We will usually write xy for $x \cdot y$. Residuated lattices give the equivalent algebraic semantics for extensions of the Full Lambek calculus (with exchange, weakening, and falsum).

Special Classes of Residuated Lattices

Residuated lattices originiate in the study of **ideal lattices of rings**. Other prominant examples from classical logic include **lattice-ordered groups** and **relation algebras**. A residuated lattice is called:

- a **GBL-algebra** if it satisfies $x(x \to y) \approx x \wedge y$.
- a **BL-algebra** if it is a GBL-algebra satisfying $(x \to y) \vee (y \to x) \approx 1$.
- a **MV-algebra** if it is a BL-algebra satisfying $\neg\neg x \approx x$, where $\neg x := x \to 0$.
- a **Heyting algebra** if it satisfies $xy \approx x \wedge y$.

These give algebraic models of the logics mentioned before.

Definition:

Let I be a set of unary function symbols. We say that an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, 0, 1, \{\Box\}_{\Box \in I})$ is an **MV(I)-algebra** provided that:

1. $(A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ is an MV-algebra.
2. For every $\Box \in I$, $\Box$ is a $\{\wedge, \cdot, 0, 1\}$-endomorphism of $(A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$.

Also:

- If $\Box$ is an interior operator for every $\Box \in I$, then we say that $\mathbf{A}$ is an **S4MV(I)-algebra**.
- An **S4MV-algebra** is an S4MV(I)-algebra where $I = \{\Box\}$ is a singleton.
- An S4MV(I)-algebra for $I = \{G, H\}$ is called a **S4$_t$MV-algebra** if for every $x, y \in A$,

$$x \leq G(y) \iff P(x) \leq y.$$

Algebraic Semantics

Theorem (F.-Zuluaga 2021):

- The variety of MV(I)-algebras is the equivalent algebraic semantics for Ł(I).
- The variety of S4MV(I)-algebras is the equivalent algebraic semantics for $S4$Ł(I).
- The variety of S4_tMV-algebras is the equivalent algebraic semantics for $S4_t$Ł.

Algebraic semantics quickly gives nice results regarding the modal logics introduced, e.g. an analysis of congruences in these algebras gives various forms of **deduction theorems** for the logics.

The Translations

Definition:

We define a pair of translations M and T from the language $\mathcal{L} = \{\wedge, \vee, \cdot, 0, 1\}$ into the languages of $S4Ł$ and $S4_tŁ$, respectively. Set $M(p) = \Box p$ for each $p \in \mathrm{Var}$, $M(0) = 0$, $M(1) = 1$, and extend M recursively by:

- $M(\varphi \star \psi) = M(\varphi) \star M(\psi)$, for $\star \in \{\wedge, \vee, \cdot\}$.
- $M(\varphi \to \psi) = \Box(M(\varphi) \to M(\psi))$.

Further, if Γ is a set of formulas of $\mathcal{L}$ then we define $M(\Gamma) = \{M(\varphi) : \varphi \in \Gamma\}$.

The translation T differs from M only by replacing $\Box$ by G and considering its codomain to be formulas of bimodal language rather than the monomodal one.

A Gödel-McKinsey-Tarski theorem

Theorem (F.-Zuluaga 2021):

Let $\Gamma \cup \{\varphi\}$ be a set of $\mathcal{L}$-formulas. Then:

1. $\Gamma \vdash_{\mathbf{GBL}} \varphi$ if and only if $M(\Gamma) \vdash_{\mathbf{S4Ł}} M(\varphi)$.
2. $\Gamma \vdash_{\mathbf{GBL}} \varphi$ if and only if $T(\Gamma) \vdash_{\mathbf{S4_tŁ}} T(\varphi)$.

The Idea of the Proof of the GMT Translation

- Proof of the GMT translation invokes algebraization along with two components.
- The first is a technical lemma regarding evaluations in GBL-algebras.
- Once one has the technical lemma, the hard part of the proof of GMT translation is showing if that $\mathbf{A}$ is a GBL-algebra, then there exists an $S4MV$-algebra $(\mathbf{B}, \Box)$ such that $\mathbf{A}$ embeds in $\mathbf{B}_\Box$.
- This second part is done by the work of Jipsen and Montagna on **poset products**.

A Key Technical Lemma

The proof of the translation proceeds algebraically, and the following is the most important lemma.

Lemma (F.-Zuluaga 2021):

Let $(\mathbf{A}, \Box)$ be an $S4MV$-algebra.

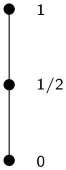
1. $\mathbf{A}_\Box$ is a GBL-algebra.
2. Suppose that $h\colon \mathrm{Var} \to \mathbf{A}$ is an assignment, and define $\bar{h}\colon \mathrm{Var} \to \mathbf{A}_\Box$ by $\bar{h}(p) = \Box(h(p))$. If $\varphi \in Fm_\mathcal{L}$, then $\bar{h}(\varphi) = h(M(\varphi))$.
3. If $\varphi \in Fm_\mathcal{L}$, then $\varphi \approx 1$ is valid $\mathbf{A}_\Box$ if and only if $M(\varphi) \approx 1$ is valid in $\mathbf{A}$.

The same holds if $\mathbf{A}$ is replaced by and $S4_tMV$ algebra, $\Box$ is replaced by G, and M is replaced by T.

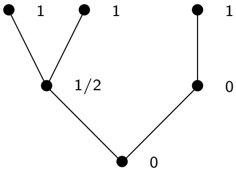
Definition:

Let $(X, \leq)$ be a poset, and let $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. An **antichain labeling** (or **ac-labeling** is a choice function $f \in \prod_{x \in X} A_x$ such that

$$\text{For all } x, y \in X \text{ with } x < y, f(x) = 0 \text{ or } f(x) = 1.$$



$\mathbf{A}_x$



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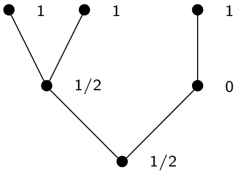

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Poset Products

Poset products are one of the most powerful tools for working with GBL-algebras.

Definition:

Let $(X, \leq)$ be a poset and let $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and greatest element 1. Set $B = \{f \in \prod_{x \in X} : f \text{ is an ac-labeling}\}$. We define operations in B as follows. The operations $\wedge, \vee, \cdot, 0, 1$ are defined pointwise, and the operation $\rightarrow$ is defined by

$$(f \rightarrow g)(x) = \begin{cases} f(x) \rightarrow_x g(x) & \text{if for all } y > x, f(y) \leq_x g(y) \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $\mathbf{B}$ with these operation is called the **poset product**.

Note: Poset products of GBL-algebras are GBL-algebas.

Conuclei and Conuclear Images

If **A** is a residuated lattice, a map $\sigma\colon A \to A$ is a **conucleus** on **A** if for all $x, y \in A$:

1. $\sigma(x) \leq x$
2. $\sigma(\sigma(x)) = \sigma(x)$.
3. $x \leq y$ implies $\sigma(x) \leq \sigma(y)$
4. $\sigma(x)\sigma(y) \leq \sigma(xy)$
5. $\sigma(x)\sigma(1) = \sigma(1)\sigma(x) = \sigma(x)$

If σ is a conucleus on $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 0, 1)$, then $\mathbf{A}_\sigma = (\sigma[A], \wedge_\sigma, \vee, \cdot, \to_\sigma, 0, \sigma(1))$ is also a residuated lattice, where $x \wedge_\sigma y = \sigma(x \wedge y)$ and $x \to_\sigma y = \sigma(x \to y)$.

Poset products as conuclear images

Let $(X, \leq)$ be a poset and $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. Set $\mathbf{B} = \prod_{x \in X} \mathbf{A}_x$ and define a map $\Box\colon B \to B$ by

$$\Box(f)(x) = \begin{cases} f(x) & \text{if } f(y) = 1 \text{ for all } y > x \\ 0 & \text{if there exists } y > x \text{ with } f(y) \neq 1. \end{cases}$$

Then $\Box$ is a conucleus on the direct product. The conuclear image coincides with the poset product:

$$\mathbf{B}_\Box = \prod_{(X,\leq)} \mathbf{A}_x.$$

AC-labelings Revisited

Actually, the following is quite easy to prove in this set up:

Lemma:

Let $f \in \mathbf{B}$ as above. The following are equivalent.

1. $f \in \mathbf{B}_{\Box}$.
2. $\Box f = f$.
3. For all $x, y \in X$ with $x < y$, $f(x) = 0$ or $f(y) = 1$.
4. $S_f = \{x \in X : f(x) \notin \{0, 1\}\}$ is a (possibly empty) antichain of $(X, \leq)$, $L_f = f^{-1}(0)$ is a down-set of $(X, \leq)$, and $U_f = f^{-1}(1)$ is an up-set of $(X, \leq)$.

Embedding Into Poset Products

In a 2010 paper, Jipsen and Montagna show tha if **A** is a GBL-algebra then there exists a poset product of MV-algebra chains into which **A** embeds.

Turns out that this poset product naturally induces an S4MV-algebra

$$(\prod_{x \in X} \mathbf{A}_x, \Box).$$

This gives the missing ingredient of the GMT translation, and turns out that the above can be upgraded to a $\mathrm{S4}_t$MV-algebra as well.

Conclusions and Future Work

- Poset products are the main engine that makes all of this work, and further analysis of poset products in the context of substructural modal logic is more than warranted.
- Temporal dimension especially merits scrutiny, and this work generalizes the work of Aguzzoli, Bianchi, and Marra on temporal semantics for Basic Logic.
- Overall deep connections to modal logic via partial canonical extensions.

Thank you!