Free Modal Riesz Spaces are Archimedean: a Syntactic Proof

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Overview

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Introduction

Merge two notions:

- Lattices $(R, \sqcup, \sqcap)$ - pervasive in logic
- Real vector spaces $(R, 0, +, (r)_{r \in \mathbb{R}})$ - pervasive in probability theory

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Well known subject:

- "Riesz spaces" (W.A.J. Luxemburg and A.C. Zaanen, 1971)
- "Introduction to Riesz Spaces" (E. De Jonge and A.C.M Van Rooij, 1977)
- "Free vector lattices" (K.A. Baker, 1968)
- "On the variety of of Riesz spaces" (C.C.A. Labuschagne and C.J. van Alten, 2007)

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- $(R, 0, +, (r)_{r\in\mathbb{R}})$ is a vector space,
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- $x \leq y \Rightarrow x + z \leq y + z$
- $\forall r \in \mathbb{R}_{\geq 0}, 0 \leq x \Rightarrow 0 \leq rx$

Important example: the real numbers is complete.

The real numbers $(\mathbb{R}, 0, +, (r)_{r\in\mathbb{R}}, \min, \max)$ is a Riesz space.
Equivalently: the real numbers with the linear order.

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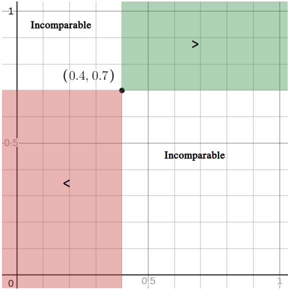
All Riesz spaces are distributive lattices.

Products of Riesz spaces

The product of Riesz spaces with pointwise operations is a Riesz space.

Example: $\mathbb{R}^2$, $\mathbb{R}^k$ and more generally function spaces $\mathbb{R}^X$

$$(a, b) + (a', b') = (a + a', b + b')$$



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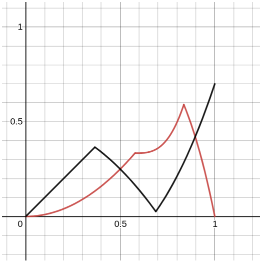

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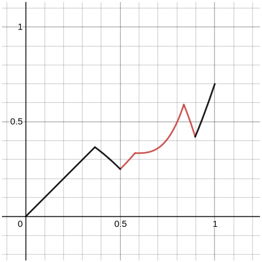
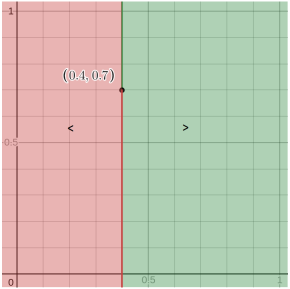


$\mathbb{R}^2$ with lexicographic order is a Riesz space.

$(a, b) \leq (c, d)$ iff $a < c$ or $a = c$ and $b \leq d$



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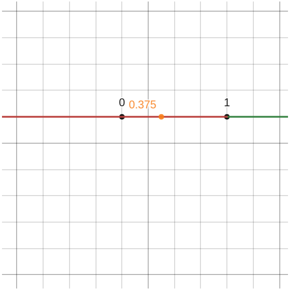

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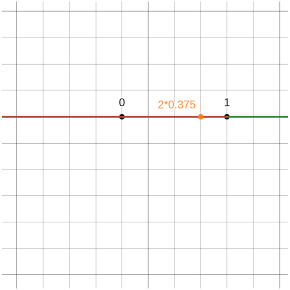

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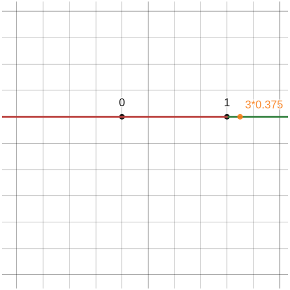
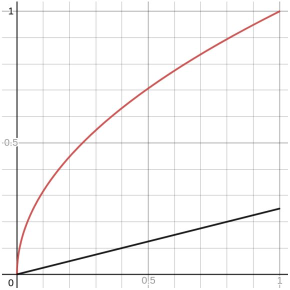


Archimedean property and function spaces

Every Riesz space $\mathbb{R}^{[0,1]}$ (with pointwise operations) is Archimedean.



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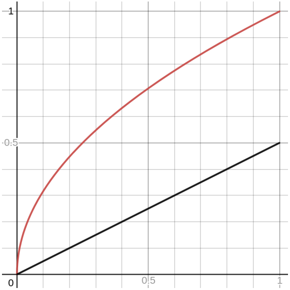

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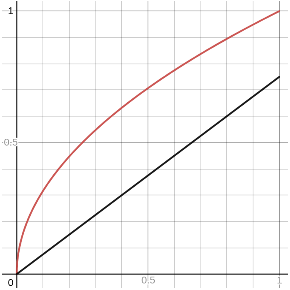

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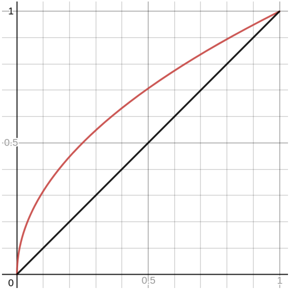

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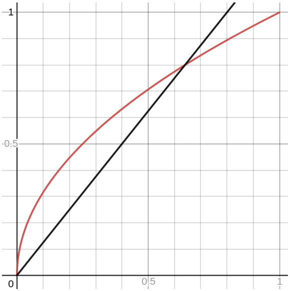
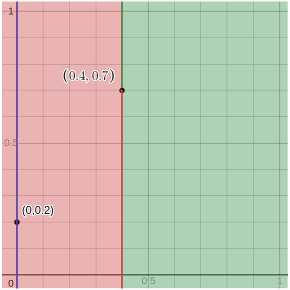


Example of a non-Archimedean Riesz space

The Riesz space $\mathbb{R}^2$ with the lexicographic order is not Archimedean:



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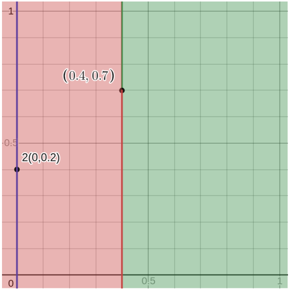

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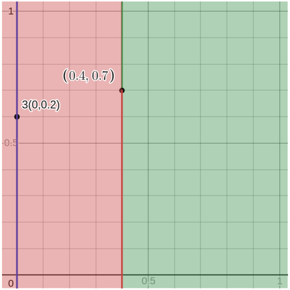

Free Riesz spaces

The free Riesz space over a set of generators V is constructed as:

1. the set of terms
   $A, B ::= x \in V \mid 0 \mid A \sqcap B \mid A \sqcup B \mid -A \mid rA \mid A + B,$
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Examples ($V = \{x_1\}$): x_1, $x_1 + 0$,...

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Corollary

Free Riesz spaces are Archimedean.

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"Probabilistic logics based on Riesz spaces" (M. Mio, R. Furber and R. Mardare, 2020)

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The modal Riesz spaces on $(\mathbb{R}, \leq)$ are exactly those equipped with $\Diamond : x \mapsto \lambda x$ with $\lambda \in [0, 1]$.

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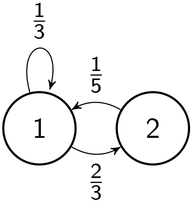
The matrix M is the transition matrix of a Markov chain.

Markov chains

Directed graphs with edges labeled by probabilities summing up to ≤ 1.



$$\Diamond = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{5} & 0 \end{pmatrix}$$

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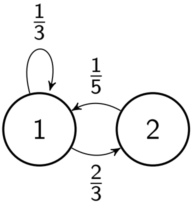

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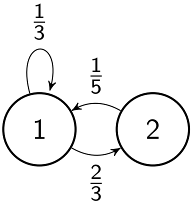

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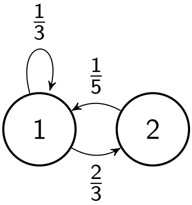

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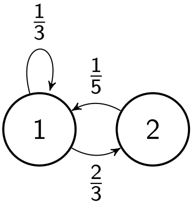

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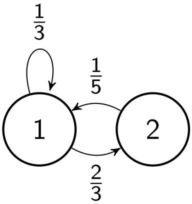

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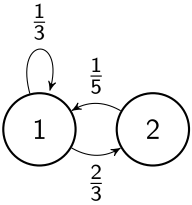

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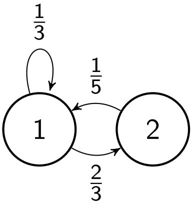

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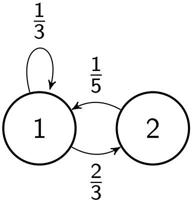

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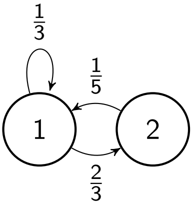

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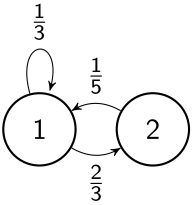
$$\Diamond \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}.1 + \frac{2}{3}.1 \\ \frac{1}{5}.1 + 0.1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{5} \end{pmatrix}$$

one more step

$$\Diamond\Diamond \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Diamond \begin{pmatrix} 1 \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{3}.1 + \frac{2}{3}.\frac{1}{5} \\ \frac{1}{5}.1 + 0.\frac{1}{5} \end{pmatrix}$$

Markov chains

Directed graphs with edges labeled by probabilities summing up to ≤ 1.



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one more step

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two more step

The real set is no longer complete

Incompleteness of $\mathbb{R}$

There exist A and B such that

- $A = B$ holds in all modal Riesz spaces $(\mathbb{R}, \lozenge)$, but
- $A = B$ does not hold in all modal Riesz spaces.

The real set is no longer complete

Incompleteness of $\mathbb{R}$

There exist A and B such that

- $A = B$ holds in all modal Riesz spaces $(\mathbb{R}, \Diamond)$, but
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$A = \Diamond(x \sqcup y)$, $B = (\Diamond x) \sqcup (\Diamond y)$.

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For each modal Riesz space $(\mathbb{R}, \Diamond)$:

- there is $\lambda \in [0, 1]$ s.t. $\Diamond : x \mapsto \lambda x$, so
- $A = \lambda \max(x, y) = \max(\lambda x, \lambda y) = B$.

The real set is no longer complete

Incompleteness of $\mathbb{R}$

There exist A and B such that

- $A = B$ holds in all modal Riesz spaces $(\mathbb{R}, \Diamond)$, but
- $A = B$ does not hold in all modal Riesz spaces.

$$A = \Diamond(x \sqcup y),\ B = (\Diamond x) \sqcup (\Diamond y).$$

Does not hold in $\mathbb{R}^2$ with:

$$\Diamond = \begin{pmatrix} \frac{2}{3} & 0 \\ \frac{1}{3} & 0 \end{pmatrix},\ x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Free modal Riesz spaces

The free modal Riesz space over a set of generators V is defined as:

1. the set of terms defined by induction:
   $A, B ::= x \in V \mid 0 \mid A \sqcap B \mid A \sqcup B \mid -A \mid rA \mid A + B \mid 1 \mid \Diamond A,$
2. quotiented by the axioms of modal Riesz spaces.

Back to the Archimedean property

Archimedean property

A modal Riesz space has the Archimedean property if

$$\forall a\ b, (\forall n, na \leq b) \Rightarrow a \leq 0$$

$(\mathbb{R}, \Diamond)$ and $(\mathbb{R}^n, \Diamond)$ are Archimedean.

Back to the Archimedean property

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Are free modal Riesz spaces Archimedean?

Back to the Archimedean property

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Are free modal Riesz spaces Archimedean?
Short answer: YES.

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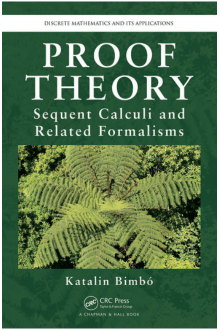

Axiom:

$$\overline{\vdash}\ \text{INIT}$$

Structural rules:

$$\frac{G}{G \mid \vdash \Gamma}\ \text{W} \qquad \frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma}\ \text{C}$$

$$\frac{G \mid \vdash \Gamma_1, \Gamma_2}{G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2}\ \text{S} \qquad \frac{G \mid \vdash \Gamma_1 \quad G \mid \vdash \Gamma_2}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{M}$$

$$\frac{G \mid \vdash r.\Gamma}{G \mid \vdash \Gamma}\ \text{T} \qquad \frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}}\ \text{ID}, \sum r_i = \sum s_i$$

Logical rules:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.0}\ 0 \quad \frac{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A+B)}\ + \quad \frac{G \mid \ \Gamma \vdash \Gamma, (s\vec{r}).A}{G \mid \ \Gamma \vdash \Gamma, \vec{r}.(sA)}\ \times$$

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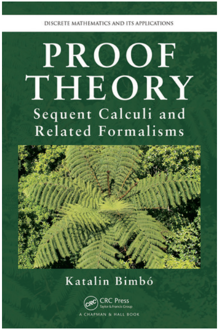
CAN rule:

$$\frac{G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma}\ \text{CAN}, \sum r_i = \sum s_i$$



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"Proof theory on Riesz spaces and modal Riesz spaces" (C. Lucas and M.Mio, LMCS, 2021)

Hypersequent calculus HMR

Sequent

A sequent is a list of pairs (r, A) where $r \in \mathbb{R}_{>0}$ and A is a term.
The sequent Γ is noted $\vdash \Gamma$.

Example: $\vdash$

$$\vdash \frac{1}{3}.(x \sqcup -x), 2.\Diamond y$$

Hypersequent calculus HMR

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Hypersequent

A hypersequent is a non–empty list of sequents. The hypersequent $(S_1, ...S_n)$ is noted $S_1 \mid ... \mid S_n$.

Example: $\vdash 2.(x + y) \mid\vdash \frac{1}{3}.\Diamond(x \sqcup -x)$

Hypersequent calculus HMR

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Derivable

We note $\triangleright_{\text{HMR}} G$ if G is derivable in the system HMR.

Hypersequents to modal Riesz spaces

Interpretation of a sequent

$$[\![\vdash r_1.A_1, ..., r_n.A_n]\!] = \sum_{i=1}^{n} r_i A_i$$

$$[\![\vdash]\!] = 0 \quad [\![\vdash \frac{1}{3}.(x \sqcup -x), 2.\Diamond y]\!] = \frac{1}{3}(x \sqcup -x) + 2\Diamond y$$

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Sound and complete

Soundness

If $\triangleright_{\mathsf{HMR}} G$ then $0 \leq [\![G]\!]$.

Completeness

If $0 \leq [\![G]\!]$ then $\triangleright_{\mathsf{HMR}} G$.

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If $0 \leq [\![G]\!]$ then $\rhd_{\mathrm{HMR}} G$.

Meaning of $\rhd_{\mathrm{HMR}}$

$\rhd_{\mathrm{HMR}} G$ if and only if $0 \leq [\![G]\!]$.

Some rules

$$\frac{}{\vdash}\ \text{INIT} \qquad\qquad \frac{G}{G \mid\vdash \Gamma}\ \text{W}$$

$$\frac{G \mid\vdash \Gamma}{G \mid\vdash \Gamma, \vec{r}.x, \vec{s}.-x}\ \text{ID}, \sum \vec{r} = \sum \vec{s} \qquad\qquad \frac{G \mid\vdash \Gamma, (s\vec{r}).A}{G \mid\vdash \Gamma, \vec{r}.(sA)}\ \times$$

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Main result: continuity of the derivability

$$G_1 = \quad \vdash \quad r_{1,1} \quad .A_1, \ldots, \quad r_{1,k} \quad .A_k \mid \ldots \mid\vdash \quad r'_{1,1} \quad .A'_1, \ldots, \quad r'_{1,k'} \quad .A_{k'}$$

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$$G_2 = \vdash r_{2,1} .A_1, \ldots, r_{2,k} .A_k \mid \ldots \mid \vdash r'_{2,1} .A'_1, \ldots, r'_{2,k'} .A_{k'}$$

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\end{array}
$$

Theorem

If $\triangleright_{\mathrm{HMR}} G_i$ for all i then $\triangleright_{\mathrm{HMR}} G$.

Corollary: the Archimedean property

Archimedean property: $(\forall n, nA \leq B) \Rightarrow A \leq 0$.

Or equivalently, $(\forall n, 0 \leq \frac{1}{n}B - A) \Rightarrow 0 \leq -A$.

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Translated to the system HMR:

$$(\forall n, \triangleright_{\text{HMR}} \vdash \frac{1}{n}.B, 1. - A) \Rightarrow \triangleright_{\text{HMR}} \vdash 1. - A$$

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Corollary

Free modal Riesz spaces are Archimedean.

One of the result used in this work

Proposition [Lemma 4.44 in our LMCS paper]

Let $G = \ \vdash \Gamma_1 \mid \ldots \mid\vdash \Gamma_n$ be an atomic hypersequent.

$$\rhd_{\mathrm{HMR}} G$$

$$\Updownarrow$$

$$\exists r_1, \ldots, r_n \in [0,1], \Big((\bigvee_{i=1}^{n} r_i = 1) \wedge \sum_{i=1}^{n} r_i.\Gamma_i = 0 \Big)$$

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$\triangleright_{\mathrm{HMR}} G = \vdash 1.x, 1.y \mid\vdash 2.-x, \mid\vdash 3.-y$ iff there are $r_1, r_2, r_3 \in [0,1]$ s.t.

- $r_1 = 1$ or $r_2 = 1$ or $r_3 = 1$, and
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$r_1 = 1, r_2 = \frac{1}{2}, r_3 = \frac{1}{3}$.

Idea of the proof

$$G_n = \ \vdash \Gamma_{1,n} \mid \ldots \mid\vdash \Gamma_{k,n} \text{ with } \Gamma_{i,n} \xrightarrow{n \to +\infty} \Gamma_i.$$

$$\forall n, \rhd_{\mathrm{HMR}} G_n$$

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$$\forall n, \triangleright_{\mathrm{HMR}} G_n \quad \Rightarrow \quad \exists t_{1,n}, \dots, t_{k,n} \in [0,1], \sum_{i=1}^{k} t_{i,n} \Gamma_{i,n} = 0$$

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Conclusion

- Probability: connection between modal Riesz spaces and Marko chains.
- Proof theory: hypersequent calculus to prove equalities on modal Riesz space terms.
- Arithmetic: reduce derivability problem to a system of polynomial equations.

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Thank you for your attention!