Polyadic spaces and profinite monoids

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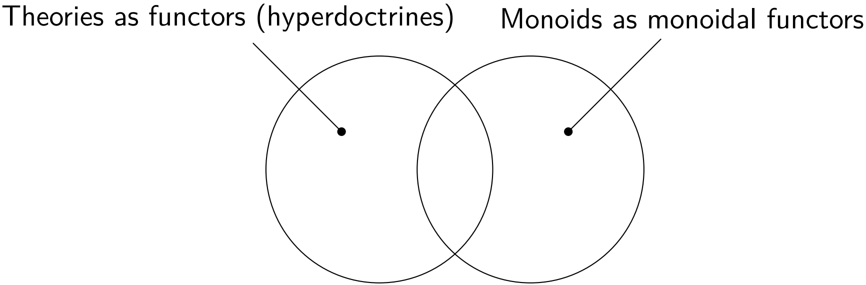
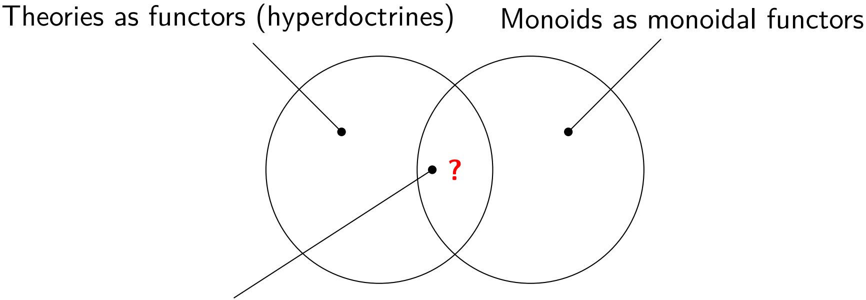
November 2, 2021
RAMiCS

Introduction



Introduction



Examples in the intersection:

- Monadic second-order logic on words $\leftrightsquigarrow$ Free profinite monoid
- First-order logic on words $\leftrightsquigarrow$ Free proaperiodic monoid

Introduction

Definition.
Type space of a 1st order theory = models modulo elementary equivalence.

Each sentence defines a subset.
The topology of the type space is generated by these subsets.
Clopens = sentences modulo provable equivalence.

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Questions:

(1) When is a profinite monoid the type space of a theory?
(2) When is the type space of a theory a profinite monoid?

Case study: first-order logic on words

Generalizing the case study

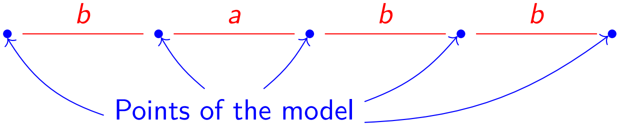
Monoids that are theories

Theories that are monoids

Case study: first-order logic on words

Case study: first-order logic on words

Intended models are finite words.



Case study: first-order logic on words

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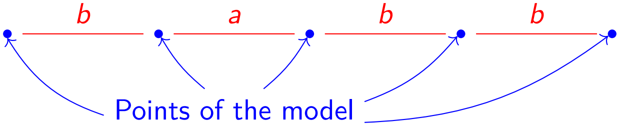

Example of sentence: "The word contains the factor ab."

$\exists p$: [the letter between p and $s(p)$ is a] $\wedge$ [the letter between $s(p)$ and $s(s(p))$ is b]

successor function

Case study: first-order logic on words

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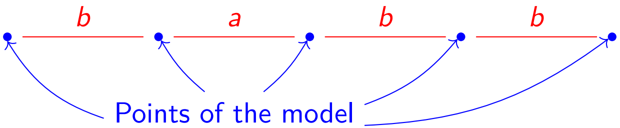

Example of sentence: "The word contains the factor *ab*."

$\exists p$: [the letter between p and $s(p)$ is a] $\wedge$ [the letter between $s(p)$ and $s(s(p))$ is b]

"The length of the word is even." cannot be expressed with a FO sentence.

Case study: first-order logic on words

FO theory of finite words $= \{$first-order $\varphi \mid \varphi$ is true on all finite words$\}$.
Pseudofinite word $W =$
a (potentially infinite) W such that $W \models \varphi$ for all $\varphi \in$ theory of finite words.

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Concatenation gives a continuous product on the type space S_0:

It is a profinite monoid.

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Theorem (Schützenberger-Pappert-McNaughton).
S_0 is the free proaperiodic monoid on the alphabet.

Case study: first-order logic on words

First-order logic on words	**Topological monoid S_0**
Sentences	Clopens
Conjunction, disjunction...	Intersection, union...
Formulas on n free variables	?
Quantifiers	?

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A pointed word = a pair of words.
Space of 1-types $S_1 = S_0^2$.

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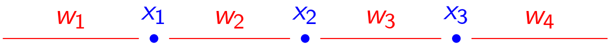
$$\begin{array}{l}\text{Space of } n\text{-types} = ? \\ \quad\quad \cup\!\mid \\ [\text{Models equipped with } n \textbf{ increasing} \text{ points } x_1 \leq \cdots \leq x_n] = S_0^{n+1}\end{array}$$

Existential quantifier = multiplication

$\varphi(x_1, x_2, x_3)$



$\in S_0^4$

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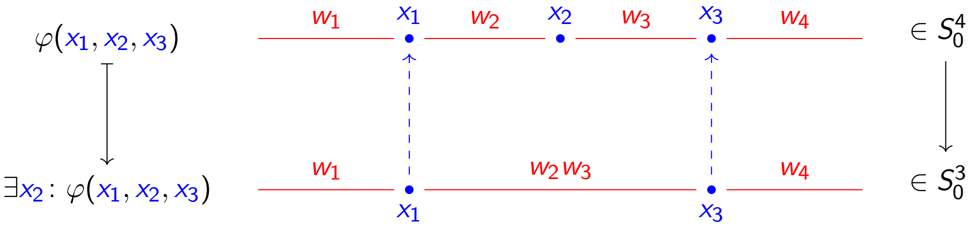
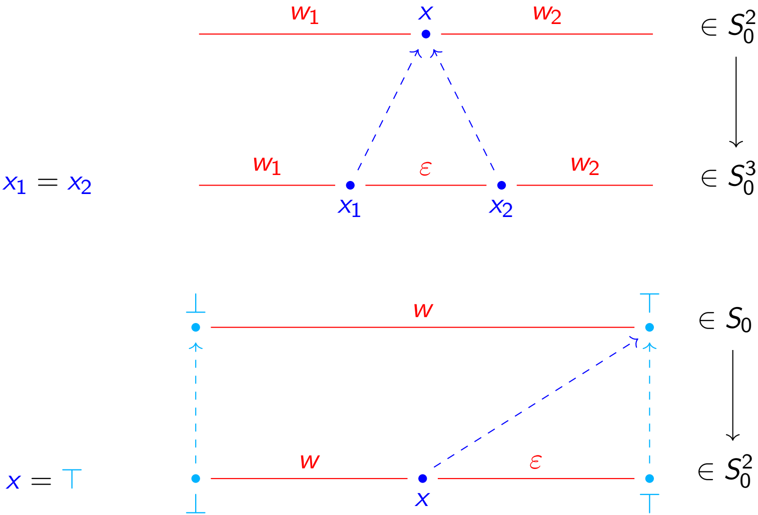


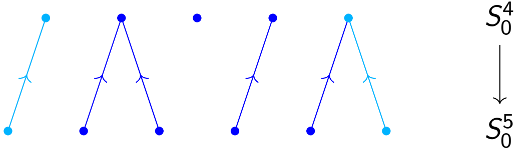
Equality and bounds = neutral element



The associated functor

Δ_{bound} = category of bounded finite linear orders

• = variables

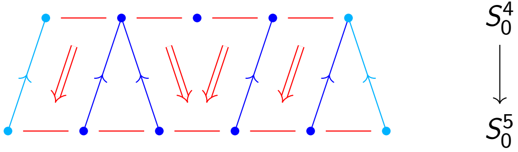


$1 + n + 1 \mapsto {S_0}^{n+1}$ is a functor $\Delta_{\mathrm{bound}}^{\mathrm{op}} \to$ Stone encoding first-order logic on words.

The associated functor

Δ_{bound} = category of bounded finite linear orders
Δ_+ = category of finite linear orders
} duals

• = variables
—— = factors



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$\wr\|$

$n \mapsto {S_0}^n$ is a functor $\Delta_+ \to$ Stone encoding the monoid S_0.

Generalizing the case study

Monoids as functors

A profinite monoid M is encoded as a functor $\Delta_+ \to \mathrm{Stone}$ sending n to M^n.
For $F : \Delta_+ \to \mathrm{Stone}$ to encode a monoid, we need a structure of *monoidal functor*:

$$F(0) = 1$$

$$\mu_{n,m} : F(n+m) \xrightarrow{\sim} F(n) \times F(m)$$

Subject to some axioms (not listed here).

Theories as functors

T = a first-order theory extending the theory of bounded linear orders.
Encoded as a functor

$$S_T : \begin{cases} \Delta^{\mathrm{op}}_{\mathrm{bound}} \to \mathsf{Stone} \\ 1 + n + 1 \mapsto \left\{ \begin{array}{l} \text{models equipped with } n \text{ increasing elements } x_1 \leq \cdots \leq x_n \\ \text{modulo elementary equivalence} \end{array} \right\} \end{cases}$$

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Definition.
We say that a functor $S : \Delta^{\mathrm{op}}_{\mathrm{bound}} \to \mathsf{Stone}$ *encodes a first-order theory* (extending the theory of bounded linear orders) if it is isomorphic to S_T for some first-order theory T.

Theories as functors

Proposition.

A functor $S : \Delta_{\text{bound}}^{\text{op}} \to \mathsf{Stone}$ *encodes a first-order theory if and only if*

(1) $S(f)$ *is open for each map* f *in* Δ_{bound}*.*

(2) The category of elements $\int S$ *has amalgamation: each span admits a cocone.*

The question

What are functors

$$\Delta_+ \cong \Delta_{\text{bound}}^{\text{op}} \to \mathsf{Stone}$$

that encode both monoids and theories?

Monoids that are theories

Monoids that are theories

Theorem 1.

Let M be a profinite monoid.
The functor $n \mapsto M^n$ encodes a first-order theory if and only if

(1) $e \in M$ is isolated and the multiplication is open, and
(2) $e \in M$ is the only invertible and M is ***equidivisible****.*

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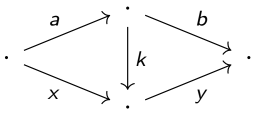
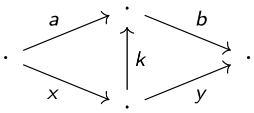
Definition.

A monoid M is *equidivisible* if for all $a, b, x, y \in M$ such that $ab = xy$, there is a $k \in M$ such that either $ak = x$ and $ky = b$, or $xk = a$ and $kb = y$.





Completeness

Let M be a profinite monoid satisfying the conditions of Theorem 1.
Let $a \in M$.
Then there is some bounded linear order X and a “distance” function

$$d : \{(x, y) \in X^2 \mid x \leq y\} \to M$$

such that...

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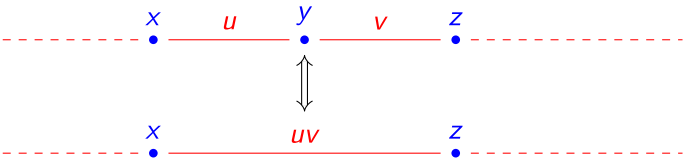
$$d : \{(x, y) \in X^2 \mid x \leq y\} \to M$$

such that

(1) $d(\min X, \max X) = a$,

(2) $d(x, y) = e \iff x = y$.

(3) $\exists y \in]x, z[: d(x, y) = u$ and $d(y, z) = v \iff d(x, z) = uv$.



Theories that are monoids

Theories that are monoids

Theorem 2.
Let T be a first-order theory extending the theory of bounded linear orders. Structures of monoidal functor on the associated functor S_T are in bijection with ***concatenation structures*** *on T.*

↑ An operation of concatenation on T-models (see next slide).

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↑ An operation of concatenation on T-models (see next slide).

Examples.

(1) Theory of all bounded linear orders.
(2) Theory of discrete bounded linear orders with letters between pairs of consecutive points.
(3) Theory of successor ordinals.
(4) Theory of dense bounded linear orders. $\leftrightsquigarrow$ Finite monoid $(\{0, 1\}, \max)$.
(5) Monadic second-order theory of finite words.

Concatenation structures on a theory

Concatenation of bounded linear orders:

Definition.
Let T be a theory extending the theory of bounded linear orders.
A *concatenation structure* on T is an associative concatenation on T-models such that:

(1) The underlying concatenation of bounded linear orders is $\boxplus$.
(2) There is only one model of size 1 which is neutral.
(3) Each pointed model (X, p) can be decomposed uniquely as

$$X = \{x \in X \mid x \leq p\} \boxplus \{x \in X \mid x \geq p\}.$$

(4) Formulas can be relativized to segments: given a formula $\varphi(\overline{x})$, there is a formula $\psi(\overline{x}, p)$ such that $(X, \overline{x}, p) \models \psi(\overline{x}, p)$ if and only if $(\{x \in X \mid x \leq p\}, \overline{x}) \models \varphi(\overline{x})$. And symmetrically with $x \geq p$ instead of $x \leq p$.