Computing Distributed Knowledge as the Greatest Lower Bound of Knowledge

RAMiCS 2021

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Motivation

- Lattices are ubiquitous in computer science.
- Join preserving endomorphisms are suitable to reason about knowledge of agents and groups of agents.
- Distributed knowledge is the knowledge that is distributed among the members of a group, without none of them necessarily having it.
- Devising (efficient) algorithms to compute distributed knowledge is complex: Exponential amount of functions.

Agenda

1. Computing the Meet of Join-Endomorphisms
2. Distributive Lattices and Knowledge Structures
3. The Distributed Knowledge Problem
4. Conclusions

Notation and Definitions

Definition (Join-endomorphisms and $\mathcal{E}(L)$)

Let L be a lattice. We say that a self-map is a *(bottom preserving) join-endomorphism* iff it preserves the join of every finite subset of L.

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- We refer to $\mathcal{E}(L)$ as the set of all join-endomorphisms of L.
- Given $f, g \in \mathcal{E}(L)$, we say $f \sqsubseteq_{\mathcal{E}} g$ iff $f(a) \sqsubseteq g(a)$ for every $a \in L$.

Roadmap

1. Computing the Meet of Join-Endomorphisms
2. Distributive Lattices and Knowledge Structures
3. The Distributed Knowledge Problem
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The Problem

Problem

Given a lattice L of size n and two join-endomorphisms $f, g : L \to L$, find the greatest join-endomorphism $h : L \to L$ below both f and g:

$$h = f \sqcap_{\mathcal{E}(L)} g.$$

Why this problem?

Note: Meet $h = f \sqcap_{\mathcal{E}(L)} g$ cannot be computed pointwise (in general):

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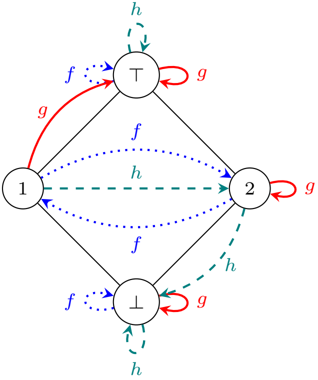

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$$h(1 \sqcup 2) \neq h(1) \sqcup h(2)$$

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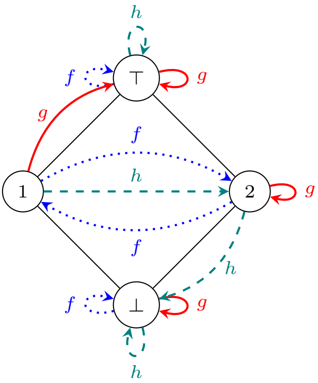

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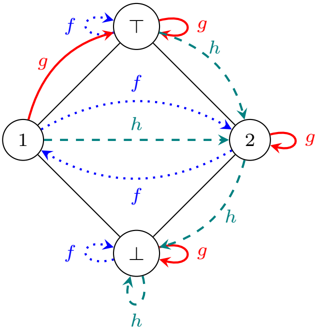
$$h(c) = \prod_L \{f(a) \sqcup g(b) \mid a \sqcup b \sqsupseteq c\}$$



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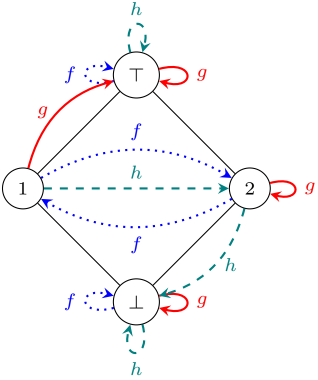

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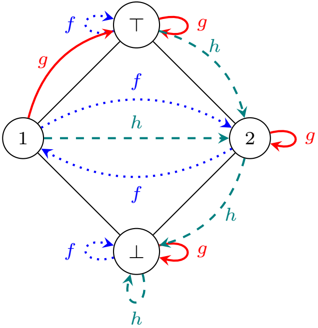

They coincide in the join-irreducible elements.

Computing $f \sqcap_{\mathcal{E}(L)} g$

Theorem

Let L be a finite distributive lattice and $f, g \in \mathcal{E}(L)$. Then $h = f \sqcap_{\mathcal{E}(L)} g$ iff h satisfies

$$h(a) = \begin{cases} f(a) \sqcap_L g(a) & \text{if } a \in \mathcal{J}(L) \text{ or } a = \bot \\ h(b) \sqcup_L h(c) & \text{if } b, c \in \downarrow^1 a \text{ with } b \neq c \end{cases}$$

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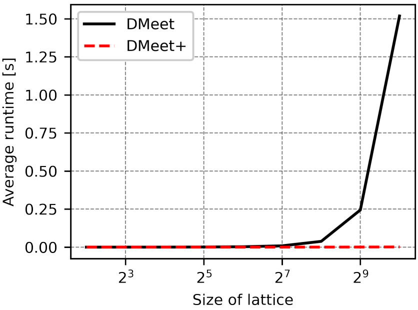
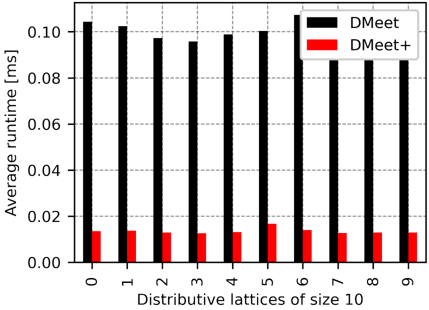
Corollary

Given a distributive lattice L of size n, and functions $f, g \in \mathcal{E}(L)$, the function $h = f \sqcap_{\mathcal{E}(L)} g$ can be computed in $O(n)$ binary lattice operations.

Experimental Results





<u>DMEET</u>

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<u>DMEET+</u>

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Representation

Lemma

Let L be a finite distributive lattice and $a, b \in \mathcal{J}(L)$. Let $f_{a,b} : L \to L$

$$f_{a,b}(x) \stackrel{\text{def}}{=} \begin{cases} b & \textit{if } x \in \uparrow a \\ \bot & \textit{otherwise} \end{cases}$$

For any join-endomorphism $f \in \mathcal{E}(L)$

$$f \textit{ is join-irreducible iff } f = f_{a,b} \textit{ for some } a, b \in \mathcal{J}(L).$$

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Corollary

Let L be a finite distributive lattice and let $f \in \mathcal{E}(L)$. Then $f = F_R$ where

$$R = \left\{(a,b) \in \mathcal{J}(L)^2 \mid a \sqsubseteq f(b)\right\} \text{ and } F_R : L \to L \text{ is}$$

$F_R(c) \stackrel{\text{def}}{=} \bigsqcup_L \{a \in \mathcal{J}(L) \mid (a,b) \in R \text{ and } c \sqsupseteq b \text{ for some } b \in \mathcal{J}(L)\}$.

Roadmap

Knowledge Structures

Definition

A (finite) *Knowledge Structure (KS)* for a set of *agents* $\mathcal{A}$ is a tuple $(\Omega, \{K_i\}_{i \in \mathcal{A}})$ where Ω is a finite set and each $K_i : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is given by

$$K_i(E) = \{\omega \in \Omega \mid \mathcal{R}_i(\omega) \subseteq E\}$$

where $\mathcal{R}_i \subseteq \Omega^2$ and $\mathcal{R}_i(\omega) = \{\omega' \mid (\omega, \omega') \in \mathcal{R}_i\}$.

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- $E \subseteq \Omega$: events
- E holds at ω iff $\omega \in E$
- K_i: knowledge operator
- $\mathcal{R}_i$: accessibility relation

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Properties:

1. $K_i(\Omega) = \Omega$
2. $K_i(E) \cap K_i(F) = K_i(E \cap F)$
3. $(K_i(E) \cap K_i(E \Rightarrow F)) \subseteq K_i(F)$
4. If $E \subseteq F$ then $K_i(E) \subseteq K_i(F)$

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4. $K_i(E) \subseteq K_i(K_i(E))$
5. $\overline{K_i(E)} \subseteq K_i\left(\overline{K_i(E)}\right)$

Extended Knowledge Structures

Definition (EKS)

A tuple $(\Omega, \mathcal{S}, \{K_i\}_{i \in \mathcal{A}})$ is said to be an *extended knowledge structure (EKS)* if

1. $(\Omega, \{K_i\}_{i \in \mathcal{A}})$ is a KS, and
2. $\mathcal{S}$ is a subset of $\mathcal{P}(\Omega)$ such that
   - $\Omega \in \mathcal{S}$
   - It is closed under union and intersection
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Notation.

- We shall use $\tilde{f}$ for the function $f\restriction_{\mathcal{S}} : \mathcal{S} \to \mathcal{P}(\Omega)$: $\tilde{f}(E) = f(E)$ for all $E \in \mathcal{S}$
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Note: Aumann Structures and, in general Knowledge Structures, are EKS where $\mathcal{S} = \mathcal{P}(\Omega)$.

Distributive lattices and EKS

Proposition

Let $(\Omega, \mathcal{S}, \{K_i\}_{i \in \mathcal{A}})$ be an EKS. Then

- $L = (\mathcal{S}, \supseteq)$ is a distributive lattice, and
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Proposition

Let L be dual to a finite lattice of sets with a family $\{f_i \in \mathcal{E}(L)\}_{i \in I}$. Then $(\Omega, \mathcal{S}, \{K_i\}_{i \in I})$ is an EKS where

- $\mathcal{S} = L$, $\Omega = \bot_L$, and
- for every $i \in I$, $\mathcal{R}_i = \{(\omega, \omega') \in \Omega^2 \mid \forall E \in \mathcal{S} : \omega \in f_i(E) \implies \omega' \in E\}$.

Furthermore, for every $i \in I$, $\widetilde{K}_i = f_i$.

Characterization

Proposition

Let L be dual to a finite powerset lattice with a family $\{f_i \in \mathcal{E}(L)\}_{i \in I}$. Let $(\Omega, \{K_i\}_{i \in I})$ be the KS where

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Corollary

Let $\mathcal{K} = (\Omega, \{K_i\}_{i \in \mathcal{A}})$ be a KS. Then

1. $\mathcal{R}_i = \left\{ (\omega, \omega') \mid \omega \in \overline{K_i\left(\overline{\{\omega'\}}\right)} \right\}$.
2. *If $\mathcal{K}$ is an AS then $\mathcal{R}_i(\omega) = \overline{K_i\left(\overline{\{\omega\}}\right)}$ for every $\omega \in \Omega$.*

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Distributed Knowledge

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Let $(\Omega, \{K_i\}_{i \in \mathcal{A}})$ be a KS and $i, j \in \mathcal{A}$. The *distributed knowledge* of i and j is represented by $D_{\{i,j\}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ defined as

$$D_{\{i,j\}}(E) = \{\omega \in \Omega \mid \mathcal{R}_i(\omega) \cap \mathcal{R}_j(\omega) \subseteq E\}$$

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Properties:

1. $(K_i(E) \cap K_j(E \Rightarrow F)) \subseteq D_{\{i,j\}}(F)$
2. $K_i(E) \subseteq D_{\{i,j\}}(E)$
3. $D_{\{i,j\}}(\Omega) = \Omega$
4. $D_{\{i,j\}}(E) \cap D_{\{i,j\}}(F) = D_{\{i,j\}}(E \cap F)$

The Meet of Knowledge

Theorem

Let $(\Omega, \mathcal{S}, \{K_i\}_{i\in\mathcal{A}})$ *be an EKS and let* L *be the lattice* $(\mathcal{S}, \supseteq)$*. Let us suppose that* $K_m = D_{\{i,j\}}$ *for some* $i, j, m \in \mathcal{A}$*. Then*

$$\widetilde{K}_m = \widetilde{K}_i \sqcap_{\mathcal{E}(L)} \widetilde{K}_j.$$

The Distributed Knowledge Problem

In what follows, let $(\Omega, \{K_i\}_{i \in \mathcal{A}})$ be a KS and let $n = |\Omega|$, and $L = (\mathcal{P}(\Omega), \supseteq)$.

Problem

Given the knowledge of agents i, j, m, decide whether m has the distributed knowledge of i and j:

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Approach 1

$K_m = D_{\{i,j\}}$ iff $K_m = K_i \sqcap_{\mathcal{E}(L)} K_j$

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$K_m = D_{\{i,j\}}$ iff $K_m = K_i \sqcap_{\mathcal{E}(L)} K_j$

Approach 2

$K_m = D_{\{i,j\}}$ iff $\mathcal{R}_m = \mathcal{R}_i \cap \mathcal{R}_j$

Approach 1

- The states in Ω are numbered as $\omega_1, \ldots, \omega_n$ where $n = |\Omega|$.
- Each event E is represented as a number $\#E \in [0..2^n - 1]$
  - Its binary representation has the k-th bit set to 1 iff $\omega_k \in E$.

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Theorem

Given the knowledge operators K_i, K_j, K_m on $\mathcal{P}(\Omega)$, there is an $O(n^2)$ algorithm for the following decision problem:

> ***Input:*** *Each K_k with $k \in \{i, j, m\}$ in array form, i.e. array $\mathtt{K}_k$ with $k \in \{i, j, m\}$ of size 2^n that stores $\#K_k(E)$ at position $\#E$ ($\mathtt{K}_k[\ \#E\ ] = \#K_k(E)$).*
>
> ***Output:*** *Boolean answer to whether $K_m = D_{\{i,j\}}$.*

Approach 2

Theorem

Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \subseteq \Omega^2$ be equivalences over a set Ω of $n = |\Omega|$ elements. There is an $O(\alpha_n n)$ algorithm for the following problem:

> ***Input:*** *Each $\mathcal{R}_i$ in partition form, i.e. an array of disjoint arrays of elements of Ω, whose concatenation produces Ω. This is readable in $O(n)$.*
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> ***Output:*** *Boolean answer to whether $\mathcal{R}_3 = \mathcal{R}_1 \cap \mathcal{R}_2$.*

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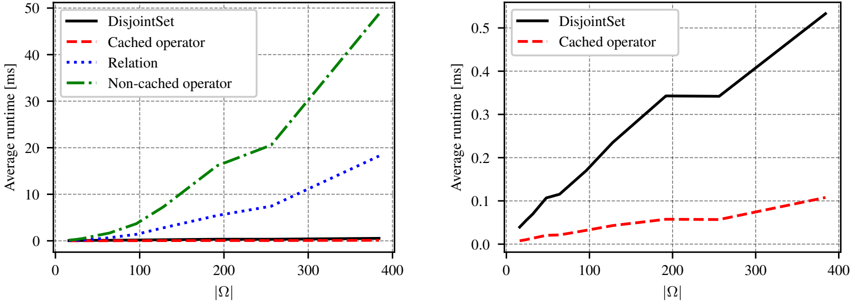
- Here α_n is the inverse of the Ackermann function:
  - $\alpha_n \stackrel{\text{def}}{=} \min\{k : A(k,k) \geq n\}$, where A is the Ackermann function.
- The growth of α_n is negligible in practice, e.g., $\alpha_n = 4$ for $n = 2^{2^{2^{65536}}} - 3$.

Experimental Results



- DisjointSet: $O(\alpha_n n)$ algorithm (Approach 2).
- Cached operator: $O(n^2)$ algorithm (Approach 1).
  - It assumes that $K_i(\cdot)$ can be evaluated in $O(1)$ at any join-irreducible.
  - Bit-mask operations are linear w.r.t. the number of bits.
- Relation: $O(n^2)$ -based on accessibility relations as $n \times n$ binary matrices.
- Non-cached operator: $O(n^2)$ -considers the cost of evaluating $K_i(\cdot)$.

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- Algorithm to compute $f \sqcap_{\mathcal{E}(L)} g$:
  - $O(n)$ for distributive lattices (previous $O(n^2)$).
  - Based on $(f \sqcap_{\mathcal{E}(L)} g)(c) = f(c) \sqcap_L g(c)$ for every $c \in \mathcal{J}(L)$

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- Algorithm to compute $f \sqcap_{\mathcal{E}(L)} g$:
  - $O(n)$ for distributive lattices (previous $O(n^2)$).
  - Based on $(f \sqcap_{\mathcal{E}(L)} g)(c) = f(c) \sqcap_L g(c)$ for every $c \in \mathcal{J}(L)$
- We have studied representation theorems for lattices.
  - Adaptation to finite distributive lattices of Jónsson-Tarski duality.

Conclusions

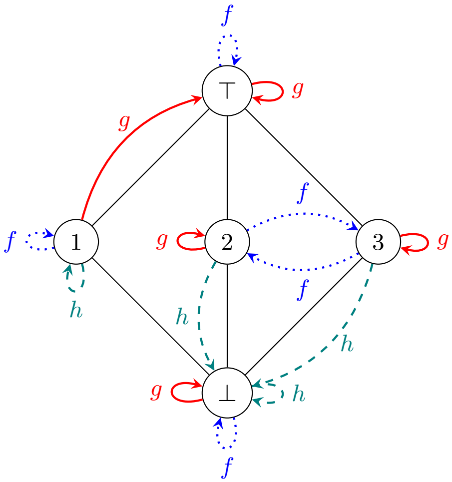
- We have used tools from lattice theory to characterize the notion of distributed knowledge.
- Algorithm to compute $f \sqcap_{\mathcal{E}(L)} g$:
  - $O(n)$ for distributive lattices (previous $O(n^2)$).
  - Based on $(f \sqcap_{\mathcal{E}(L)} g)(c) = f(c) \sqcap_L g(c)$ for every $c \in \mathcal{J}(L)$
- We have studied representation theorems for lattices.
  - Adaptation to finite distributive lattices of Jónsson-Tarski duality.
- We have provided an algorithm to compute the intersection of *partitions* of a set of size n in $O(n\alpha_n)$.

Thanks

Thanks

Questions?

Counterexample: Non-distributive Case



Any $h : \mathbf{M}_3 \to \mathbf{M}_3$ s.t. $h(a) = f(a) \sqcap g(a)$ for $a \in \mathcal{J}(\mathbf{M}_3)$ is not in $\mathcal{E}(\mathbf{M}_3)$:

$$h(\top) = h(1 \sqcup 2) = h(1) \sqcup h(2) = 1 \neq \bot = h(2) \sqcup h(3) = h(2 \sqcup 3) = h(\top).$$

Experimental Results

Size	DMEET Time [s]	DMEET+ Time [s]	DMEET #$\sqcup$	DMEET+ #$\sqcup$	DMEET #$\sqcap$	DMEET+ #$\sqcap$
16	0.000246	0.000024	81	11	81	4
32	0.000971	0.000059	243	26	243	5
64	0.002659	0.000094	729	57	729	6
128	0.008735	0.000163	2187	120	2187	7
256	0.038086	0.000302	6561	247	6561	8
512	0.244304	0.000645	19683	502	19683	9
1024	1.518173	0.001468	59049	1013	59049	10

Average runtime in seconds over powerset lattices. Number of $\sqcup$ and $\sqcap$ operations performed for each algorithm.