ℓr-Multisemigroups, Modal Quantales and the Origin of Locality

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Motivation

a quantale $(Q, \leq, \cdot, 1)$ is a complete lattice $(Q, \leq)$ and a monoid $(Q, \cdot, 1)$, and $\cdot$ preserves sups in both arguments for $f, g : X \to Q$ from relational structure $(X, R)$ with ternary $R$ a convolution operation is defined as

$$(f \ast g)(x) = \bigvee_{R(x, y, z)} f(y) \cdot g(z)$$

the convolution algebra is the algebra on $Q^X$
Convolution as a binary modality

\[(f \ast g)(x) = \bigvee_{R(x,y,z)} f(y) \cdot g(z) \quad f, g : X \to Q\]

in Lambek calculus, convolution is a binary modality over a ternary frame

in boolean algebras with operators

\(n\)-ary modalities in \(B\) are dual to \(n + 1\)-ary relations in \(X\)

there are correspondences between properties in \(X\) and identities in \(B\)
for general $Q$ we get correspondence triangles

$$
\begin{array}{c}
Q^X \\
\downarrow \\
X \\
\downarrow \\
Q
\end{array}
$$

these yield uniform \textit{construction recipes} for convolution algebras $Q^X$
from $X$ and value algebras $Q$

today: find $X$ corresponding to \textit{modal quantale} $B^X$, with view on locality
Modal quantales

A domain quantale is a quantale with $\text{dom} : Q \to Q$ satisfying

$$\text{dom}(\alpha) \cdot \alpha = \alpha$$  \hspace{1cm} \text{(absorption)}
$$\text{dom}(\alpha \lor \beta) = \text{dom}(\alpha) \lor \text{dom}(\beta)$$  \hspace{1cm} \text{(finite sup-preservation)}
$$\text{dom}(\bot) = \bot$$  \hspace{1cm} \text{(empty sup-preservation)}
$$\text{dom}(\alpha) \leq 1$$  \hspace{1cm} \text{(subidentity)}
$$\text{dom}(\alpha \cdot \beta) = \text{dom}(\alpha \cdot \text{dom}(\beta))$$  \hspace{1cm} \text{(locality)}

A codomain quantale $(Q, \text{cod})$ is a domain quantale in $(Q^{op}, \text{dom})$

A modal quantale is a domain and codomain quantale such that

$$\text{dom} \circ \text{cod} = \text{cod} \hspace{1cm} \text{cod} \circ \text{dom} = \text{dom}$$  \hspace{1cm} \text{(compatibility)}

If $B^X$ is a modal quantale ... which structure has $X$?
Object-free category?

categories. A category is a set $C$ of arrows with two functions $s, t : C \to C$, called "source" and target", and a partially defined binary operation $\#$, called composition, all subject to the following axioms, for all $x, y, z$ in $C$:

The operation $x \# y$ is defined iff $sx = ty$ and then

$$s(x \# y) = sy, \quad t(x \# y) = tx;$$  \hspace{1cm} (1)

$$x \# sx = x, \quad tx \# x = x;$$  \hspace{1cm} (2)

$$(x \# y) \# z = x \# (y \# z) \quad \text{if either side is defined;}$$  \hspace{1cm} (3)

$$ssx = sx = tsx;$$ \hspace{1cm} (4)

$$ttx = tx = stx.$$  \hspace{1cm} (4)

Then $x$ is an identity iff $x = sx$ or, equivalently, iff $x = tx.$

[MacLane, Ch.XII.5]

we want to be more general/relational
Multioperations

\[ \mathcal{P}(X \times X \times X) \cong X \times X \to \mathcal{P}X \]

a multioperation is a function \( X \times X \to \mathcal{P}X \)

we extend \( \odot : X \times X \to \mathcal{P}X \) to

\[ \odot : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X, \ (A, B) \mapsto \bigcup \{x \odot y \mid x \in A, y \in B\} \]

\( \odot \) is a partial operation if \( |x \odot y| \leq 1 \) for all \( x, y \in X \)

\( \odot \) is a (total) operation if \( |x \odot y| = 1 \) for all \( x, y \in X \)

the shuffle of words is a multioperation
\( \ell r \)-Multisemigroups

an \( \ell r \)-multimagma is a multimagma \((X, \odot)\) with \( \ell, r : X \to X \) satisfying

\[
x \odot y \neq \emptyset \Rightarrow r(x) = \ell(y)
\]

\[
\ell(x) \odot x = \{x\} \quad x \odot r(x) = \{x\}
\]

(absorption)

an \( \ell r \)-multisemigroup is an associative \( \ell r \)-multimagma

\[
x \odot (y \odot z) = (x \odot y) \odot z
\]

an \( \ell r \)-multimagma is \( \ell r \)-local if \( r(x) = \ell(y) \Rightarrow x \odot y \neq \emptyset \)

categories are precisely the partial \( \ell r \)-local \( \ell r \)-semigroups

\( \ell r \)-locality captures the composition pattern of categories
Examples

pair groupoid $(A \times A, \odot, \ell, r)$ is an $\ell r$-local partial $\ell r$-semigroup with

$$(a, b) \odot (c, d) = \begin{cases} \{ (a, d) \} & \text{if } b = c \\ \emptyset & \text{otherwise} \end{cases} \quad \ell((a, b)) = (a, a) = r((b, a))$$

shuffle $\ell r$-multisemigroup $(A^*, \|, \ell, r)$ is $\ell r$-local because $\|$ is total with $\ell(w) = \varepsilon = r(w)$

PAMs with unit 1 used in separation logic are non-local because $\ell(x) = 1 = r(x)$ and composition is partial

paths $f : [0, 1] \rightarrow T$ in topology form local partial $\ell r$-magmas

more generally, elements of $X_\ell = \{ x \mid \ell(x) = x \} = \{ x \mid r(x) = x \} = X_r$ are orthogonal idempotent units of $X$ ($\ell(x) \neq \ell(y) \iff \ell(x) \ell(y) = \emptyset$)
Properties

in $\ell r$-multimagmas

$$
\ell \circ r = r \quad r \circ \ell = \ell \quad \text{(compatibility)}
$$

$$
\ell(\ell(x)y) = \ell(x)\ell(y) \quad r(xr(y)) = r(x)r(y) \quad \text{(export)}
$$

in $\ell r$-multisemigroups

$$
\ell(xy) \subseteq \ell(x\ell(y)) \quad r(xy) \subseteq r(r(x)y) \quad \text{(weak locality)}
$$

in $\ell r$-local $\ell r$-multisemigroups

$$
\ell(xy) = \ell(x\ell(y)) \quad r(xy) = r(r(x)y) \quad \text{(locality)}
$$
Origin of locality

\( \ell r \)-multisemigroup is \( \ell r \)-local iff

\[
\ell(x\ell(y)) = \ell(xy) \quad r(r(x)y) = r(xy)
\]

equational locality thus captures the composition pattern of categories

\( xy \neq \emptyset \iff r(x) = \ell(y) \)
Constructing powerset quantales

if \((X, \odot, \ell, r)\) is an \(\ell r\)-multisemigroup, then \((\mathcal{P}X, \subseteq, \odot, X_\ell)\) is a boolean quantale whose underlying lattice is atomic

all \(\ell(x)/r(x)\) in \(X\) are combined into unit \(X_\ell\) of \(\mathcal{P}X\)

categories lift to powerset quantales with arrows as atoms

pair groupoids lift to quantales of binary relations

we refine this construction lifting \(\text{dom} = \mathcal{P}\ell\) and \(\text{cod} = \mathcal{P}r\)
Lifting to modal powerset quantales

from $\ell r$-multimagma

\[
\ell(A)A = A \\
\ell(A) \cup \bigcup_{A \in \mathcal{A}} \ell(A) = \bigcup_{A \in \mathcal{A}} \ell(A) \\
\ell(A) \subseteq X_{\ell} \\
\ell(r(A)) = r(A) \\
\ell(\ell(A)B) = \ell(A)\ell(B)
\]

$Ar(A) = A$  \hspace{1cm} (absorption)

\[
r(A) \subseteq X_{r} \\
r(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} r(A) \\
r(l(A)) = l(A) \\
r(\ell(\ell(A)B)) = r(A)r(\ell(B))
\]

$\ell$-$r$-preservation

\[
\ell \left( \bigcup_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} \ell(A) \\
r \left( \bigcup_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} r(A)
\]

(sub-preservation)

\[
\ell(A) \subseteq X_{\ell} \\
r(A) \subseteq X_{r}
\]

(subidentity)

\[
\ell(r(A)) = r(A) \\
r(\ell(A)) = \ell(A)
\]

(compatibility)

\[
\ell(\ell(A)B) = \ell(A)\ell(B) \\
r(\ell(\ell(A)B)) = r(A)r(\ell(B))
\]

(export)

from $\ell r$-multisemigroup

\[
\ell(AB) \subseteq \ell(Al(B)) \\
r(AB) \subseteq r(r(A)B)
\]

(weak locality)

from local $\ell r$-multisemigroup

\[
\ell(AB) = \ell(Al(B)) \\
r(AB) = r(r(A)B)
\]

(locality)
Modal correspondences

If $X$ is an $\ell r$-multimagma,
- then $(\mathcal{P}X, \subseteq, \odot, X_\ell, \text{dom}, \text{cod})$ is a boolean modal prequantale
- it is a weakly local modal quantale if $X$ is an $\ell r$-multisemigroup
- it is a modal quantale if $X$ is a local $\ell r$-multisemigroup

We get converse directions, too
- if $\mathcal{P}X$ is a prequantale, then $X$ is an $\ell r$-multimagma
- if $\mathcal{P}X$ is a quantale, then $X$ is an $\ell r$-multisemigroup
- if $\mathcal{P}X$ is a modal quantale, then $X$ is a local $\ell r$-multisemigroup

We don't see $xy \neq \emptyset \iff r(x) = \ell(y)$ in quantale instead we see $\alpha \beta \neq 0 \iff \text{cod}(\alpha)\text{dom}(\beta) \neq 0$
Modal Correspondences

\[ \text{dom}(A \odot \text{dom}(B)) = \bigcup \{ \ell(x \odot \ell(y)) \mid x \in A, y \in B, r(x) = \ell(\ell(y)) \} \]
\[ = \bigcup \{ \ell(x \odot y) \mid x \in A, y \in B, r(x) = \ell(y) \} \]
\[ = \text{dom}(A \odot B) \]

\[ \ell(x \odot \ell(y)) = \text{dom}({x} \odot \text{dom}({y})) \]
\[ = \text{dom}({x} \odot {y}) \]
\[ = \ell(x \odot y) \]
Examples

categories lift to modal powerset quantales

pair groupoids lift to modal powerset quantales of binary relations

shuffle multisemigroups lift to quantales of shuffle languages

PAMs lift to non-local assertion quantales of separation logic

path algebras in topology lift to modal powerset prequantales

Jónsson/Tarski knew that groupoids lift to RAs with converse
Discussion

further examples can be found in paper

results extend to convolution algebras \( Q^X \) with correspondence triangles

\( Q \) can be semiring/Kleene algebra when finiteness properties for \( \lor \) hold

see arXiv paper for details
we introduced $\ell r$-multisemigroups
related them with categories
showed how $\ell / r$ correspond to $\text{dom/cod}$
explained how locality relates to composition pattern of categories
presented generic construction recipe for modal powerset quantales