Unary-Determined Distributive ℓ-magmas and Bunched Implication Algebras

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Motivation: Structural Description of Idempotent Semirings

Idempotent semirings $(A, \vee, \cdot)$ play an important role in several areas of computer science including network optimization, formal languages, and program semantics.

Semirings are called **(additively) idempotent** if $x + x = x$ and **doubly idempotent** if $x \cdot x = x$ as well. In this case $+$ and $\cdot$ define a join semilattice and a meet semilattice respectively.

In [1] a complete structural description was given for finite commutative doubly idempotent semirings where either the multiplicative semilattice is a chain, or the additive semilattice is a Boolean algebra.

We can generalize the second description significantly, to the setting where the the additive semilattice is a distributive lattice and the multiplication satisfies the condition $x \cdot y = (px \wedge y) \vee (x \wedge qy)$ for a pair of unary operators p and q.

Distributive lattice-ordered magmas

Definition

A **distributive lattice-ordered magma** ($d\ell$-**magma** for short) $(A, \wedge, \vee, \bot, \top, \cdot)$ is a bounded distributive lattice with a binary operation $\cdot$ such that for all $x, y, z \in A$

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$\wedge$-free reducts of $d\ell$-monoids are **(additively) idempotent semirings**.

Complete and completely join-preserving $d\ell$-monoids are **unital quantales** and they expand uniquely to **complete distributive residuated lattices**.

Unary-determined $d\ell$-magmas

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$$x \wedge \neg x = \bot \quad \text{and} \quad x \vee \neg x = \top.$$

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Theorem

Every idempotent Boolean magma is unary-determined.

Term-equivalence for unary-determined $d\ell$-magmas

Definition

A $d\ell pq$-**algebra** $(A, \wedge, \vee, \bot, \top, p, q)$ is a bounded distributive lattice with two unary operations p, q that satisfy

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\begin{array}{lll}
p\bot = \bot & p(x \vee y) = px \vee py & x \wedge p\top \leq qx \\
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1. *Let* A *be a* $d\ell pq$*-algebra and define* $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$*. Then* $(A, \wedge, \vee, \bot, \top, \cdot)$ *is a* $d\ell$*-magma that is unary-determined and* p, q *are definable as* $px = x{\cdot}\top$ *and* $qx = \top{\cdot}x$*.*

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2. *Let* $\mathbf{A}$ *be a unary-determined* $d\ell$*-magma and define* $px = x \cdot \top$, $qx = \top \cdot x$. *Then* $(A, \wedge, \vee, \bot, \top, p, q)$ *is a* $d\ell pq$*-algebra and* $\cdot$ *is definable as* $x \cdot y = (px \wedge y) \vee (x \wedge qy)$.

Associativity, Commutativity, Idempotence from p, q

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Let $(A, \wedge, \vee, \bot, \top, p, q)$ *be a* $d\ell pq$*-algebra and* $x \cdot y = (px \wedge y) \vee (x \wedge qy)$.

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5. *The operation* $\cdot$ *has an identity* 1 *if and only if* $p1 = \top = q1$ *and* $(px \vee qx) \wedge 1 \leq x$.

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6. *If* $\cdot$ *has an identity then* $\cdot$ *is idempotent.*

Heyting algebras and bunched implication algebras

Definition

A **Heyting algebra** $(A, \wedge, \vee, \bot, \top, \to)$ is a bounded lattice $(A, \wedge, \vee, \bot, \top)$ such that $\to$ is the residual of $\wedge$, i. e.,

$$x \wedge y \leq z \quad \iff \quad y \leq x \to z.$$

The residual $\to$ ensures that the lattice is **distributive**.

Definition

A **bunched implication algebra** (BI-algebra) $(A, \wedge, \vee, \bot, \top, \to, *, 1, -\!*)$ is a Heyting algebra $(A, \wedge, \vee, \bot, \top, \to)$ such that $(A, *, 1)$ is a commutative monoid and $-\!*$ is the residual of $*$, i. e.,

$$x * y \leq z \quad \iff \quad y \leq x -\!* z.$$

The class of Heyting algebras and BI-algebras can both be defined by equations, so they are **varieties**.

BI-algebras and BI-logic

Bunched implication algebras are the algebraic semantics of **BI-logic**

BI-logic is the propositional part of **separation logic**, which is a Hoare logic for reasoning about data structures and memory allocation.

The structure of BI-algebras is not well understood.

Defining $\neg x = x \to \bot$ and adding $\neg\neg x = x$ to BI-algebras gives the variety of Boolean BI-algebras, which contains the variety CRA of commutative relation algebras.

Finite BI-algebras "=" finite commutative distributive residuated lattices.

Every BI-algebra has a commutative $d\ell$-monoid as a reduct.

Every finite commutative $d\ell$-monoid expands uniquely to a BI-algebra.

Aim: find easy-to-describe subvarieties of BI-algebras

A BI-algebra is **idempotent** if $x*x = x$.

Recall that a **preorder** P is a binary relation that is reflexive and transitive

Theorem (Alpay, Jipsen 2020)

Every finite idempotent Boolean *BI-algebra is determined by a preorder P on the set of atoms such that*

P is a preorder forest: xPy and xPz implies yPz or zPy, and

P has singleton roots: xPy and yPx and $\forall z(xPz \implies zPx)$ implies $x = y$

Preorder forests with singleton roots are counted by an Euler transform:

n	1	2	3	4	5	6	7	8	9	10	11
f_n	1	2	5	14	41	127	402	1306	4314	14465	49054
idem. BBI f_n	1	1	0	2	0	0	0	5	0	0	0

Term equivalence for idemp. unary-determined BI-algebras

An operation p^* is the **residual** of p if $px \le y \iff x \le p^* y$ holds.

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Theorem

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   Define $x{*}y = (px \wedge y) \vee (x \wedge py)$ *and* $x{-\!*}y = (px \to y) \wedge p^*(x \wedge y)$*.*

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   Then $(A, \wedge, \vee, \bot, \top, \to, *, -\!*, 1)$ *is an idempotent unary-determined BI-algebra.*

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2. *Let* $(A, \wedge, \vee, \top, \bot, \to, *, -\!*, 1)$ *be an idempotent unary-determined BI-algebra, and define* $px = \top{*}x$ *and* $p^*x = \top{-\!*}x$.

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   Then $(A, \wedge, \vee, \to, \top, \bot, p, p^*, 1)$ *is a Heyting algebra with an operation p that has p^* as residual and satisfies* $x \leq px = ppx$, $px \wedge py \leq p((px \wedge y) \vee (x \vee py))$, $p1 = \top$ *and* $px \wedge 1 \leq x$.

Downsets and completely join-irreducibles

Definition

Let $(W, \leq)$ be a poset. A **downset** is a subset $X \subseteq W$ such that $y \leq x \in X$ implies $y \in X$.

- Let $D(W, \leq)$ be the set of all downsets.
- The **downset lattice** is $(D(W, \leq), \cap, \cup, \emptyset, W)$.
- The downset lattice is a bounded distributive lattice.

Definition

An element x in a lattice A is **completely join-irreducible** if

$$x \neq \bigvee \{y \in A \mid y < x\}.$$

$J(A)$ denotes the set of completely join-irreducibles of A.

Kripke semantics for finite dℓ-magmas

Definition

$(J(A), \leq, R)$ is the **Birkhoff frame** of a finite $d\ell$-magma A with the ternary relation R defined by $R(x, y, z) \iff x \leq y{\cdot}z$.

- From $(x \vee y){\cdot}z = x{\cdot}z \vee y{\cdot}z$ it follows that $\cdot$ is order preserving.
- Hence R satisfies:

(R1) $u \leq x \;\&\; R(x, y, z) \implies R(u, y, z)$ (downward closure)
(R2) $R(x, y, z) \;\&\; y \leq v \implies R(x, v, z)$ (upward closure)
(R3) $R(x, y, z) \;\&\; z \leq w \implies R(x, y, w)$ (upward closure).

Definition

In general a **Birkhoff frame** $\mathrm{W} = (W, \leq, R)$ is a poset $(W, \leq)$ with a ternary relation $R \subseteq W^3$ that satisfies (R1),(R2),(R3).

Birkhoff frames produce $d\ell$-magmas

Definition

For a Birkhoff frame W define the **downset algebra** $D(\mathrm{W}) = (D(W, \leq), \cap, \cup, \cdot, \emptyset, W)$, where for $Y, Z \in D(W, \leq)$

$$Y \cdot Z = \{x \in W \mid R(x, y, z) \text{ for some } y \in Y \text{ and } z \in Z\}.$$

$Y \cdot Z$ is a downset by (R1),(R2),(R3) of R.

Theorem

Let W *be a Birkhoff frame. Then*

- $D(\mathrm{W})$ *is a* $d\ell$*-magma.*
- $D(\mathrm{W})$ *is idempotent if and only if for all* $x, y, z \in W$, $R(x, x, x)$, *and* $(R(x, y, z) \implies x \leq y \text{ or } x \leq z)$.

PQ-Frames and P-Frames

Definition

$(W, \leq, P, Q)$ is a **PQ-frame** if

1. $(W, \leq)$ is a poset.
2. $u \leq x \;\&\; P(x, y) \;\&\; y \leq v \implies P(u, v)$
3. $u \leq x \;\&\; Q(x, y) \;\&\; y \leq v \implies Q(u, v)$

i.e., P, Q are **weakening relations**.

- A **P-frame** is a PQ-frame where $P = Q$.
- P is **reflexive** if $P(x, x)$ for all $x \in W$.
- P is **transitive** if $P(x, y) \;\&\; P(y, z) \implies P(x, z)$.

Note: $x \leq y \;\&\; P(y, y) \implies P(x, y)$ by weakening.

We can define unary operations p and q on $D(W)$ by $pY = \{x \in W \mid \exists y \in Y\ (xPy)\}$ for a downset Y, with q defined analogously. Then the downset algebra is a $d\ell pq$-algebra.

Correspondence theory for PQ-Frames and $d\ell pq$-algebras

Lemma

Let $\mathsf{W} = (W, \leq, P, Q)$ *be a PQ-frame, and* $\mathsf{A} = D(\mathsf{W})$ *a* $d\ell pq$*-algebra. If it exists, the constant* $1 \in A$ *corresponds to a downset* $E \subseteq W$*. Then*

1. $x \leq px$ *holds in* A *if and only if* P *is reflexive,*
2. $ppx \leq px$ *holds in* A *if and only if* P *is transitive,*
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3. $px = qx$ *holds in* A *if and only if* $P = Q$*,*
4. $p1 = \top$ *holds in* A *if and only if* $\forall x \exists y (y \in E \;\&\; xPy)$ *holds in* W*,*

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1. $x \leq px$ *holds in* A *if and only if* P *is reflexive,*
2. $ppx \leq px$ *holds in* A *if and only if* P *is transitive,*
3. $px = qx$ *holds in* A *if and only if* $P = Q$*,*
4. $p1 = \top$ *holds in* A *if and only if* $\forall x \exists y (y \in E \;\&\; xPy)$ *holds in* W*,*
5. $px \wedge 1 \leq x$ *holds in* A *if and only if* $x \in E \;\&\; xPy \Rightarrow x \leq y$ *in* W*,*

Correspondence theory for PQ-Frames and $d\ell pq$-algebras

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6. $px \wedge py \leq p((px \wedge y) \vee (x \wedge py))$ *holds in* $\mathbf{A}$ *if and only if* $wPx \ \&\ wPy \Rightarrow \exists v (wPv \ \&\ (vPx \ \&\ v \leq y \text{ or } v \leq x \ \&\ vPy))$ *in* $\mathbf{W}$*.*

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If $x \leq px = ppx$ then from (6) we get associativity in the term-equivalent $d\ell$-magma $(A, \wedge, \vee, \bot, \top, \cdot)$, where $x \cdot y = (px \wedge y) \vee (x \wedge py)$.

Preorder forest P-frames

Definition

$(W, \leq, P)$ is a **preorder-forest** P**-frame** if

1. $(W, \leq)$ is a poset.
2. $u \leq x \;\&\; xPy \;\&\; y \leq v \implies uPv$
3. xPx for all $x \in W$
4. xPy and $yPz \implies xPz$
5. xPy and $xPz \implies x \leq y$ or $x \leq z$ or yPz or zPy (Pforest)

That is $(W, \leq, P)$ is a P-frame in which P is a preorder (i. e. reflexive and transitive) and satisfies the preorder forest formula

$$\text{(Pforest)} \quad xPy \text{ and } xPz \implies x \leq y \text{ or } x \leq z \text{ or } yPz \text{ or } zPy.$$

Main Result

Theorem

Let $\mathsf{W} = (W, \leq, P)$ *be a preorder forest P-frame and* $D(\mathsf{W})$ *its corresponding downset algebra. Then the operation* $x*y = (px \wedge y) \vee (x \wedge py)$ *is associative in* $D(\mathsf{W})$.

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Note that this is **not** an equivalence:

Let $W = \{0, 1, 2, 3\}$ and $\leq\, = id_W \cup \{(0, 1), (0, 2), (0, 3)\}$. Let $P = \,\leq \cup \{(1, 0), (1, 2), (1, 3)\}$. Then $(W, \leq, P)$ is a P-frame with associative $*$, but in which (Pforest) fails.

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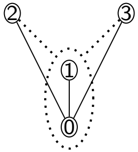

Figure: $(W, \leq, P)$ represented as a Hasse diagram with solid lines representing $\leq$ and dotted lines representing P.

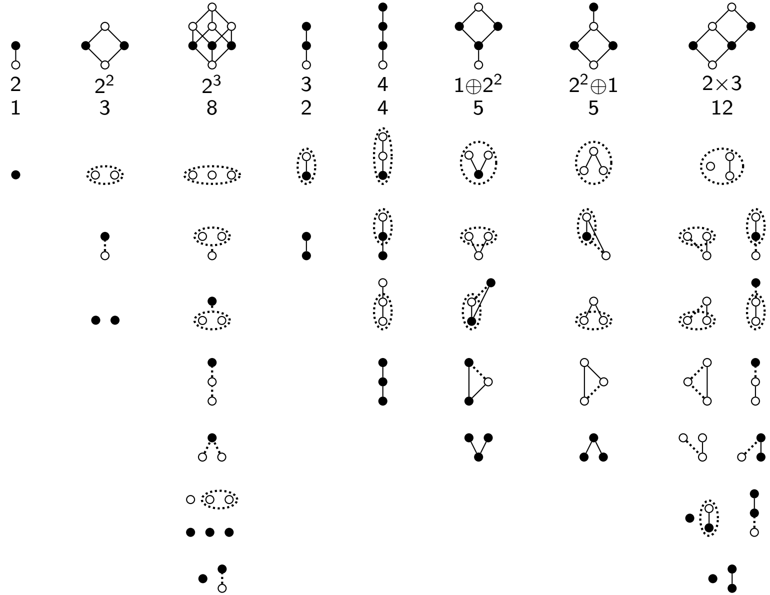

Figure: All 40 preorder forest P-frames $(W, \leq, P)$ with up to 3 join-irreducibles. Solid lines show $(W, \leq)$, dotted lines show the additional edges of P, and the identity (if it exists) is the set of black dots. The first row shows the lattice of downsets.

A closer look

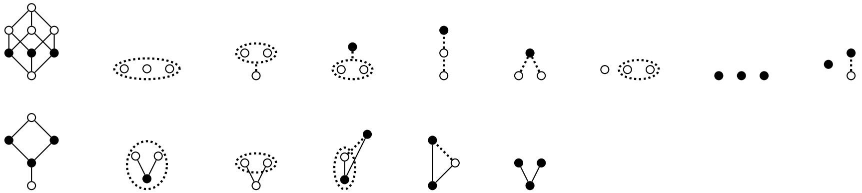

Figure: An excerpt from the previous table. The first row represents all the preorder forest P-frames $(W, \leq, P)$ whose lattice of downsets $D(W, \leq)$ forms the Boolean cube. In particular, these are Boolean quantales from [1]. However, we also represent new algebras, such as those in the second row, whose downset lattice is the "lollipop" distributive lattice with five elements.

Conclusion

- Distributive lattices with unary operations are simpler than ones with binary operations. Hence the term-equivalence between unary-determined $d\ell$-magmas and $d\ell pq$-algebras is useful.
- We defined Birkhoff frames for $d\ell$-magmas, and PQ-frames for $d\ell pq$-algebras. These frames are logarithmic in size compared to the algebras.
- Preorder forest P-frames can be calculated more efficiently than idempotent unary-determined BI-algebras, and the P-frames can be drawn as Hasse diagrams of the poset (solid lines) and the preorder (dotted and solid lines).

n	1	2	3	4	5	6	7	8	9
all BI f_n	1	1	3	16	70	399	2261		
idem. BI f_n	1	1	2	6	15	44	115	326	
idem. u-d BI f_n	1	1	2	5	10	24	47	108	223

Open problems

Do commutative idempotent unary-determined $d\ell$-monoids have a decidable equational theory?

Do idempotent unary-determined BI-algebras have a decidable equational theory?

Find an axiomatization for the variety generated by residuated complex algebras of preorder forest P-frames.

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