Quantified Constraint Satisfaction Problem: towards the classification of complexity

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Quantified Equality Constraints

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$\mathrm{QCSP}(\mathbb{N}; x = y)$

Given a sentence $\forall x_1 \exists x_2 \dots \forall x_{n-1} \exists x_n \ (x_{i_1} = x_{j_1} \wedge \dots \wedge x_{i_s} = x_{j_s})$.
Decide whether it holds.

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QCSP($\mathbb{N}; R$)

Given a sentence $\forall x_1 \exists x_2 \ldots \forall x_{n-1} \exists x_n \, (R(\ldots) \wedge \cdots \wedge R(\ldots))$.
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Accessible to anyone

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- QCSP($\mathbb{N}; x = y \rightarrow y = z$) is coNP-hard [Bodirsky, Chen, 2010].

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What is the complexity of QCSP($\mathbb{N}; x = y \rightarrow y = z$)?

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Lemma [Zhuk, Martin, 2021]

QCSP($\mathbb{N}; x = y \rightarrow y = z$) is PSpace-hard.

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What is the complexity of QCSP($\mathbb{N}; x = y \to y = z$)?

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Lemma [Zhuk, Martin, 2021]

QCSP($\mathbb{N}; x = y \to y = z$) is PSpace-hard.

Theorem [Zhuk, Martin, Bodirsky, Chen, 2021]

Suppose relations $R_1, \dots, R_s$ are definable by some Boolean combination of atoms of the form $(x = y)$. Then QCSP($\mathbb{N}; R_1, \dots, R_s$) is either tractable, NP-complete, or PSpace-complete.

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A is a finite set,
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Examples:
$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$.

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Main Question

What is the complexity of QCSP(Γ) for different Γ?

Σ	dual-Σ	Classification	Complexity Classes

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$\{\exists, \forall, \wedge\}$			

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$\{\exists, \vee\}$	$\{\forall, \wedge\}$	Trivial	L

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where $R_1, \ldots, R_s \in \Gamma$.
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Given a sentence $\exists y_1 \ldots \exists y_t ((R_1(\ldots) \vee R_2(\ldots)) \wedge R_3(\ldots))$, where $R_1, \ldots, R_3 \in \Gamma$.
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Given a sentence
$\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t ((\neg R_1(\ldots) \vee R_2(\ldots)) \wedge \neg R_3(\ldots))$,
where $R_1, \ldots, R_3 \in \Gamma$.
Decide whether it holds.

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CSP(Γ):

Given a formula $(R_1(\dots) \wedge \dots \wedge R_s(\dots))$,
where $R_1, \dots, R_s \in \Gamma$.
Decide whether the formula is satisfiable.

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if for all $\begin{pmatrix} a_1^1 \\ \vdots \\ a_1^s \end{pmatrix}, \dots, \begin{pmatrix} a_n^1 \\ \vdots \\ a_n^s \end{pmatrix} \in R$,

$$f \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^s & \dots & a_n^s \end{pmatrix} = \begin{pmatrix} f(a_1^1, \dots, a_n^1) \\ \vdots \\ f(a_1^s, \dots, a_n^s) \end{pmatrix} \in R$$

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(equivalently, f is a polymorphism of R, shortly $f \in \mathrm{Pol}(R)$)

if for all $\begin{pmatrix} a_1^1 \\ \vdots \\ a_1^s \end{pmatrix}, \dots, \begin{pmatrix} a_n^1 \\ \vdots \\ a_n^s \end{pmatrix} \in R$,

$$f \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^s & \dots & a_n^s \end{pmatrix} = \begin{pmatrix} f(a_1^1, \dots, a_n^1) \\ \vdots \\ f(a_1^s, \dots, a_n^s) \end{pmatrix} \in R$$

f preserves Γ (equivalently $f \in \mathrm{Pol}(\Gamma)$) if f preserves every $R \in \Gamma$.

Constraint Satisfaction Problem

A is a finite set,
Γ is a set of relations on A (a constraint language)

CSP(Γ):

Given a sentence $\exists y_1 \dots \exists y_t (R_1(\dots) \wedge \dots \wedge R_s(\dots))$,
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Theorem [Bulatov, Zhuk, 2017]

- CSP(Γ) is solvable in polynomial time (tractable) if there exists a weak near-unanimity operation preserving Γ,
- CSP(Γ) is NP-complete otherwise.

Weak near-unanimity operation (WNU) is an operation satisfying

$$w(y, x, x, \dots, x) = w(x, y, x, \dots, x) = \dots = w(x, x, \dots, x, y)$$

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Examples: $x \vee y, x \wedge y, xy \vee xz \vee yz, x + y + z, 0, \min(x, y), \dots$

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Are there any other complexity classes?

QCSP classifications

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- Boolean structures. Dichotomy P, Pspace-complete.
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Surjective polymorphisms

Observation

Suppose each relation of Γ_1 is definable from Γ_2 using quantified conjunctive formulas

$$R(x_1, \ldots, x_n) = \forall y_1 \exists y_2 \forall y_3 \exists y_4 \ldots R_1(\ldots) \wedge \cdots \wedge R_s(\ldots).$$

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Then $\mathrm{QCSP}(\Gamma_1)$ is polynomially reducible to $\mathrm{QCSP}(\Gamma_2)$.

Theorem (Galois Correspondence, Börner, Bulatov, Chen, Jeavons, and Krokhin, 2003)

Γ_1 *is definable by quantified conjunctive formulas over* Γ_2 *IFF each surjective polymorphism of* Γ_2 *is a polymorphism of* Γ_1*.*

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- The complexity of QCSP(Γ) depends only on surjective polymorphisms of Γ.

Surjective polymorphisms

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Suppose each relation of Γ_1 is definable from Γ_2 using primitive positive formulas

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Two questions

- What makes QCSP(Γ) easy?
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Π_2-restriction of QCSP.

$\text{QCSP}^{\Pi_2}(\Gamma)$:

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- How many tuples is it sufficient to check?

PGP vs EGP

For an algebra $(A; F)$ (a set of operations F on a set A)
$d_F(n)$ is the minimal size of a generating set of A^n.

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Examples

1. $A = \{0, 1\}$, $F = \{x \vee y\}$. $d_F(n) = n + 1$. It is sufficient to have $(0, \ldots, 0)$ and $(0, \ldots, 0, 1, 0, \ldots, 0)$ for any position of 1 to generate $\{0, 1\}^n$.

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Pair (a_i, a_{i+1}) with $a_i \neq a_{i+1}$ is a switch in a tuple $(a_1, \ldots, a_n)$.
$(0, 0, 0, 1, 2, 2, 0, 0, 0, 0)$ has 3 switches,
$(3, 3, 3, 4, 3, 3, 3, 3, 3, 3)$ has 2 switches.

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Theorem[Zhuk, 2015]

A finite algebra $\mathbf{A}$ has PGP IFF there exists k such that each $\mathbf{A}^n$ is generated by all tuples with at most k switches.

From Π_2 to NP

$\mathrm{QCSP}^{\Pi_2}(\Gamma)$:

Given a sentence $\forall x_1 \dots \forall x_t \exists y_1 \dots \exists y_q (R_1(\dots) \wedge \dots \wedge R_s(\dots))$, where $R_1, \dots, R_s \in \Gamma$.
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Example

If $x \vee y$ preserves Γ then it is sufficient to check that $(R_1(\dots) \wedge \dots \wedge R_s(\dots))$ is satisfiable for $(x_1, \dots, x_t) = (0, \dots, 0)$ and $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_t) = (0, \dots, 0, 1, 0, \dots, 0)$ for $\forall i$.

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Observation

If $\mathrm{Pol}(\Gamma)$ has PGP, then $\mathrm{QCSP}^{\Pi_2}(\Gamma)$ can be polynomially reduced to $\mathrm{CSP}(\Gamma \cup \{x = a \mid a \in A\})$.

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Proof:

From Π_2 to NP

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If $x \vee y$ preserves Γ then it is sufficient to check that $(R_1(\dots) \wedge \dots \wedge R_s(\dots))$ is satisfiable for $(x_1, \dots, x_t) = (0, \dots, 0)$ and $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_t) = (0, \dots, 0, 1, 0, \dots, 0)$ for $\forall i$.

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If Pol(Γ) has PGP, then $QCSP^{\Pi_2}(\Gamma)$ can be polynomially reduced to CSP($\Gamma \cup \{x = a \mid a \in A\}$).

Proof: the instance is equivalent to the CSP instance

$$\bigwedge_{\substack{(a_1,\dots,a_t) \text{ with} \\ \text{at most } k \text{ switches}}} (R_1(\dots) \wedge \dots \wedge R_s(\dots) \wedge (x_1 = a_1) \wedge \dots \wedge (x_t = a_t))$$

From PSpace to NP

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$$\exists y \forall x \ \Phi$$

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$$\exists y \forall x\ \Phi$$
$$\Updownarrow$$
$$\forall x^1 \forall x^2 \ldots \forall x^{|A|} \exists y\ \Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_{|A|}$$

- Φ_i is obtained from Φ by renaming x by x^i

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$$\forall x_1^1 \ldots \forall x_1^{|A|} \; \forall x_2^1 \ldots \forall x_2^{|A|^2} \ldots \forall x_t^1 \ldots \forall x_t^{|A|^t}$$
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- We keep variables with the switches

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- We keep variables with the switches
- We assign $x_1^1 = \cdots = x_1^{|A|} = 1, \dots, x_t^1 = \cdots = x_t^{|A|^t} = 0$

From PSpace to NP

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Theorem

Suppose $\mathrm{Pol}(\Gamma)$ has PGP. Then $\mathrm{QCSP}(\Gamma)$ is polynomially reducible to $\mathrm{CSP}(\Gamma \cup \{x = a \mid a \in A\})$.

From PSpace to NP

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Suppose $\mathrm{Pol}(\Gamma)$ has PGP. Then $\mathrm{QCSP}(\Gamma)$ is polynomially reducible to $\mathrm{CSP}(\Gamma \cup \{x = a \mid a \in A\})$.

* For Γ containing all constants relations this was shown earlier by Chen, Martin, Carvalho, and Madelaine.

From PSpace to NP

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Suppose $\mathrm{Pol}(\Gamma)$ has PGP. Then $\mathrm{QCSP}(\Gamma)$ is in NP.

Corollary 2

Suppose $\mathrm{Pol}(\Gamma)$ has PGP. Then $\mathrm{QCSP}(\Gamma)$ is either tractable, or NP-complete.

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Suppose Γ contains $\{x = a \mid a \in A\}$. Then QCSP(Γ)

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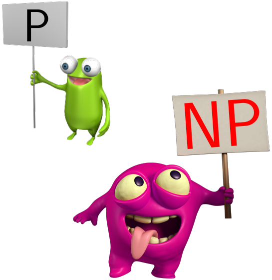

Chen Conjecture

Chen Conjecture (QCSP Trichotomy Conjecture)

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Weak Chen Conjecture

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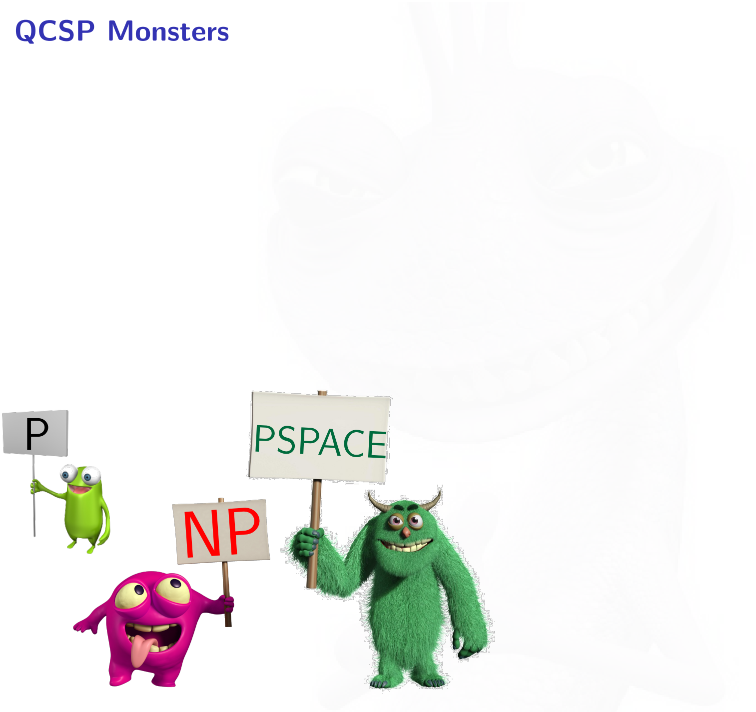
Lemma (Classification for the conservative case) [Zhuk, Martin, 2018]

Chen Conjecture holds for Γ containing all unary relations.

QCSP Monsters



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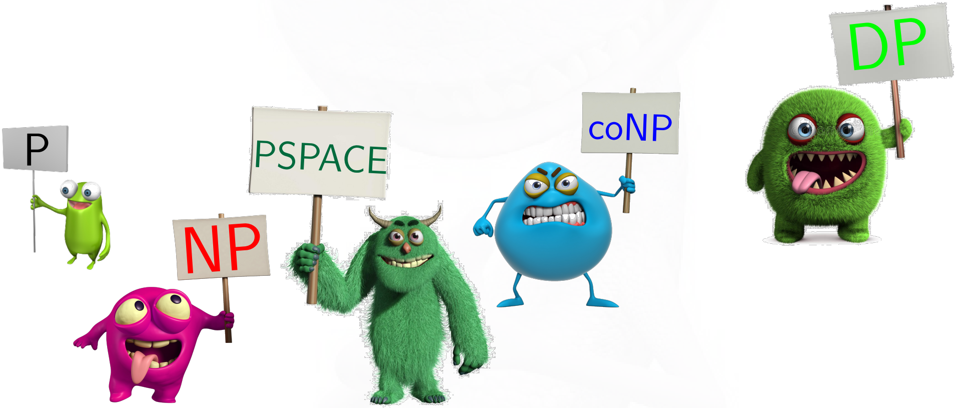

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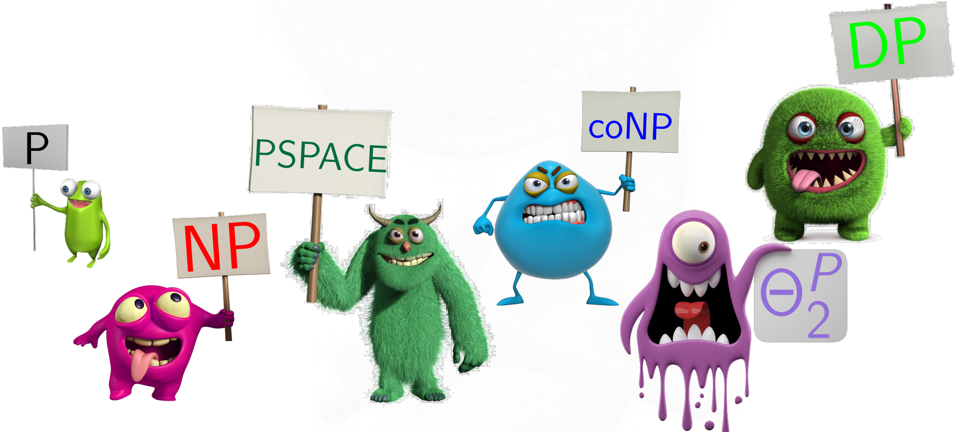

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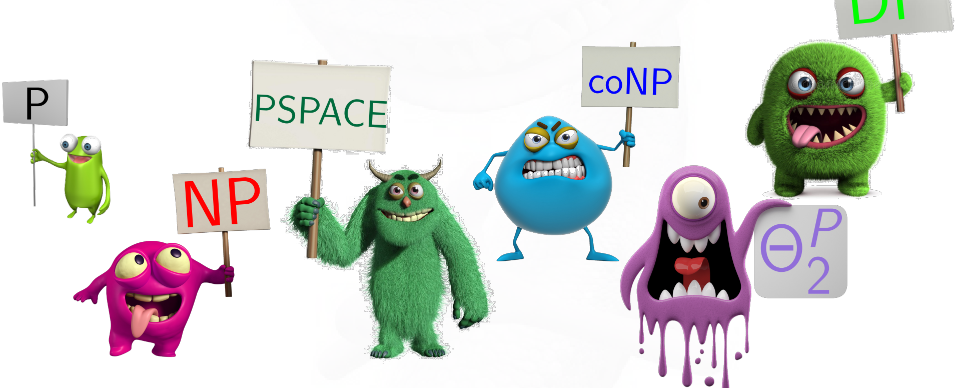

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Are there any other monsters???

Classification for a 3-element-domain

Classification for a 3-element-domain

Theorem (Classification for a 3-element domain)

Suppose Γ is a constraint language on $\{0, 1, 2\}$ containing $\{x = a \mid a \in \{0, 1, 2\}\}$. Then QCSP($\Gamma$) is

- in P, or
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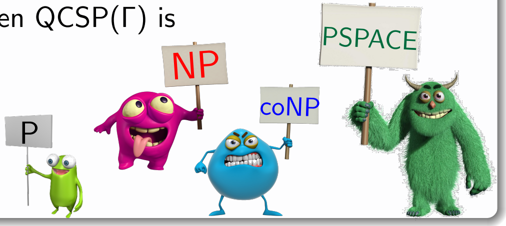

Two questions

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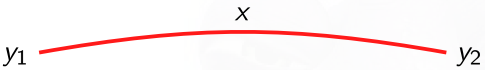

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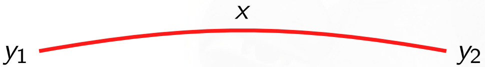

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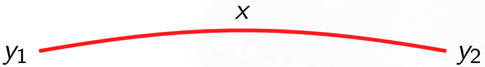

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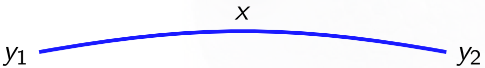

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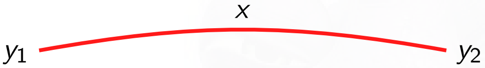

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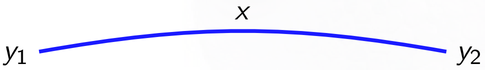
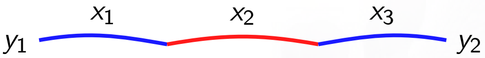


$$\exists u_1 \exists u_2 R_1(y_1, u_1, x_1) \wedge R_0(u_1, u_2, x_2) \wedge R_1(u_2, y_2, x_3)$$



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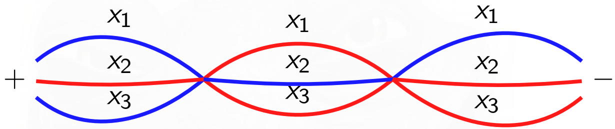

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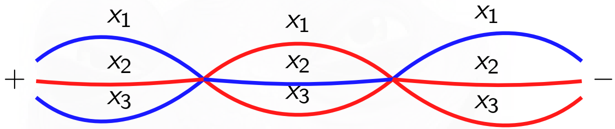

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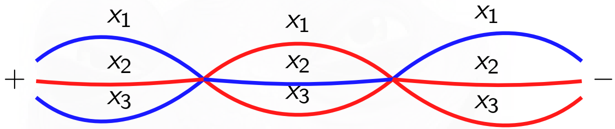

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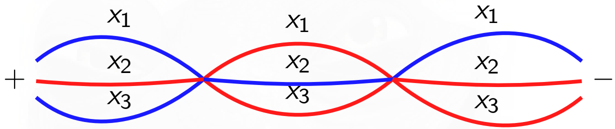

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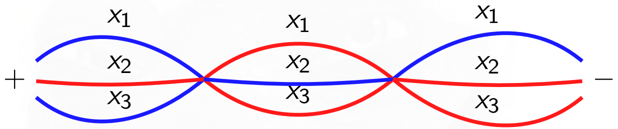
$\forall x_1 \forall x_2 \forall x_3$ 

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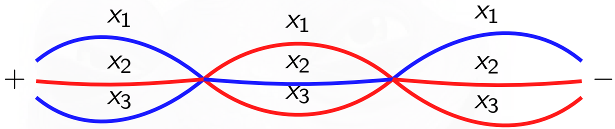

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Claim

QCSP(Γ) is coNP-hard.

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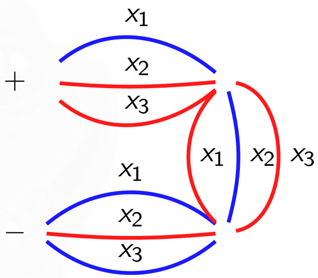

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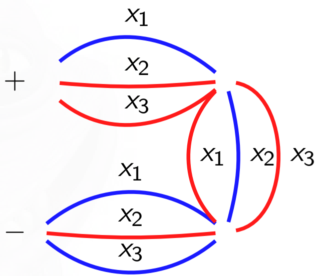

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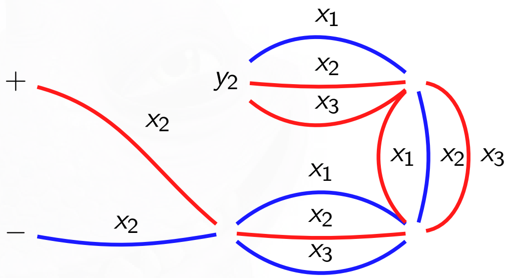

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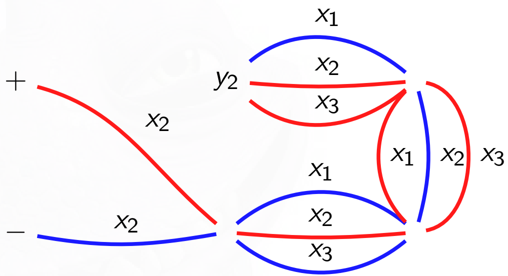

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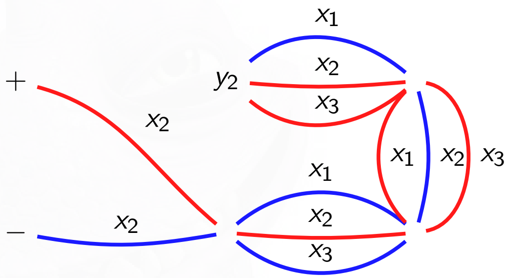

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Claim

QCSP(Γ) is PSpace-hard.

Theorem (Π_2^P vs PSpace)

Suppose Γ *contains* $\{x = a \mid a \in A\}$. *Then* QCSP(Γ)

- *is PSpace-hard if there exists a reflexive relation* $S \subsetneq A^n$ *and a nontrivial equivalence relation* σ *on* $D \subseteq A$ *such that* $R(y_1, y_2, x_1, \ldots, x_n) = \sigma(y_1, y_2) \vee S(x_1, \ldots, x_n)$ *is definable by a positive primitive formula over* Γ
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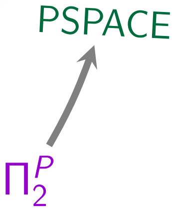

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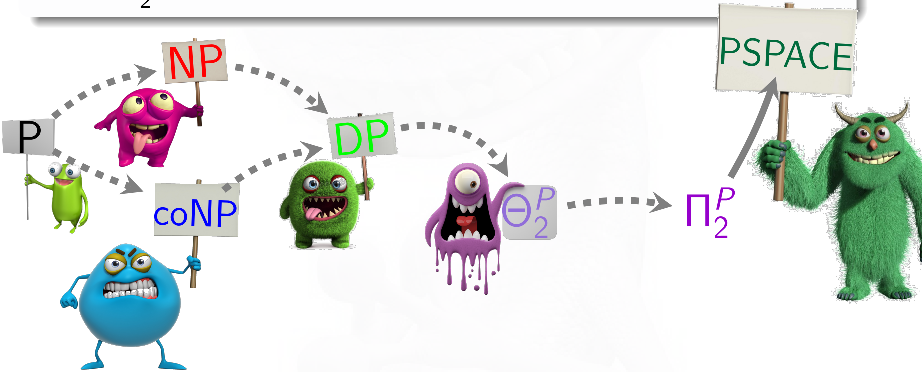

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- *in* Π_2^P *otherwise.*

Theorem (Π_2^P vs PSpace)

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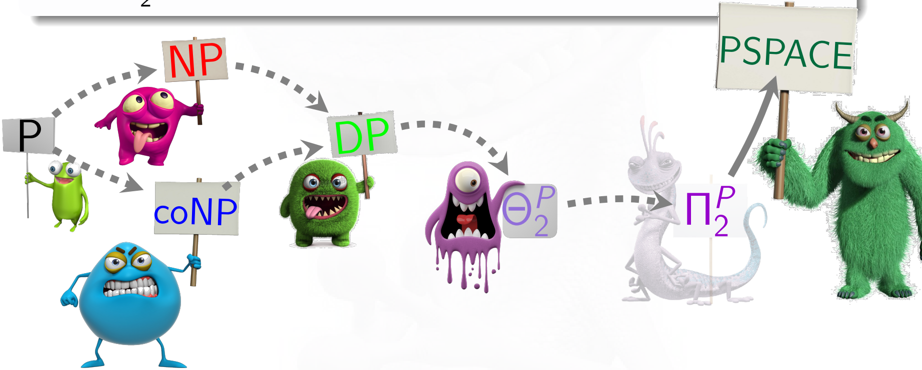

Theorem (Π_2^P vs PSpace)

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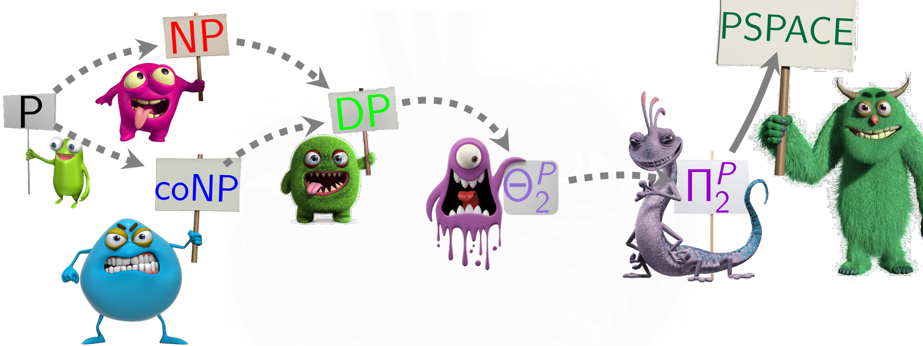
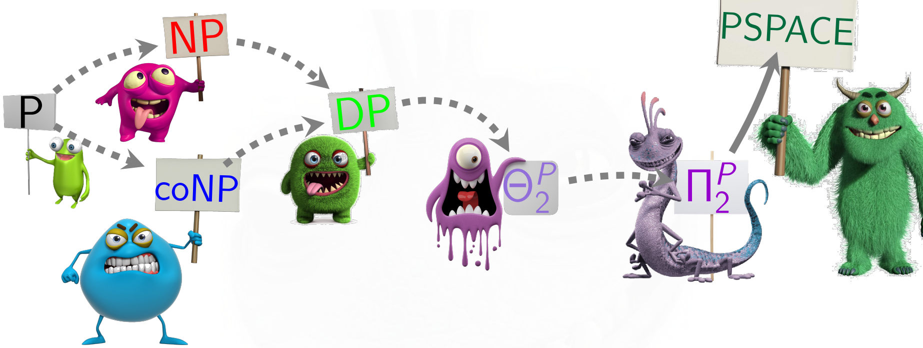
- *in* Π_2^P *otherwise.*

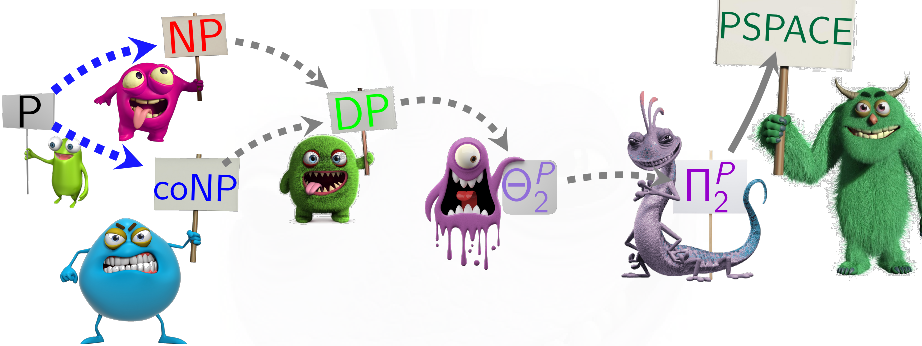
Lemma

There exists Γ on a 6-element set such that QCSP(Γ) is Π_2^P-complete.


P
NP
coNP
DP
Θ_2^P
Π_2^P
PSPACE

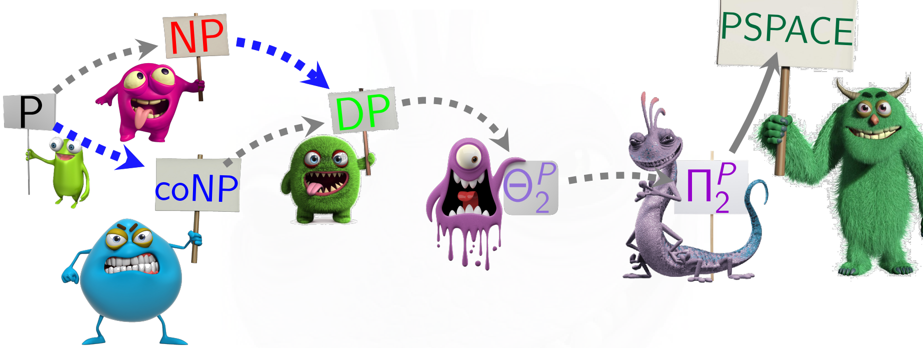


QCSP Hepta-chotomy to prove



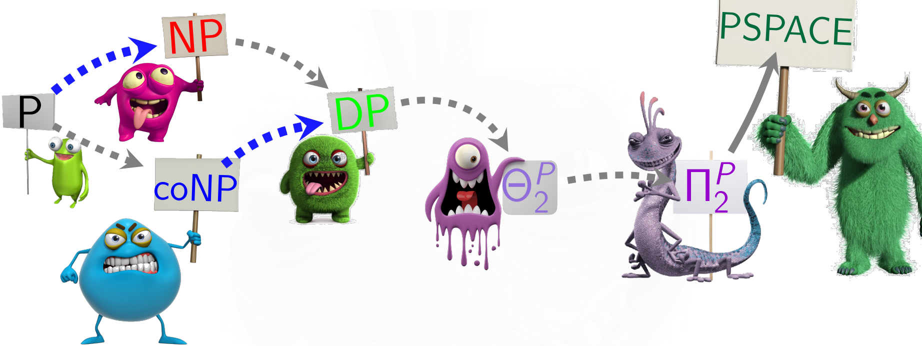
QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).



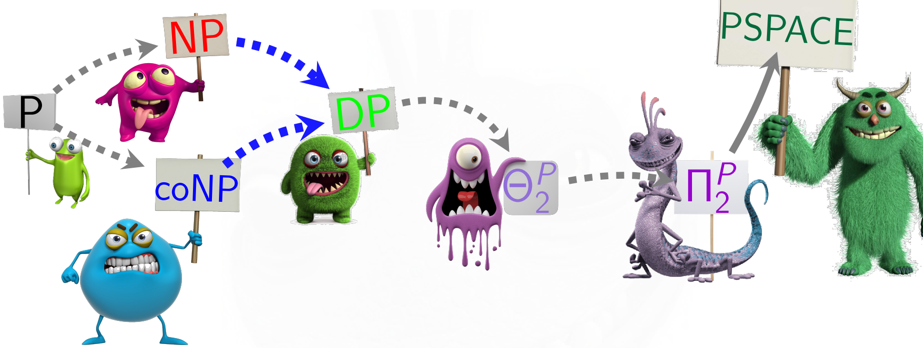
QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard



QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard
3. coNP vs NP-hard

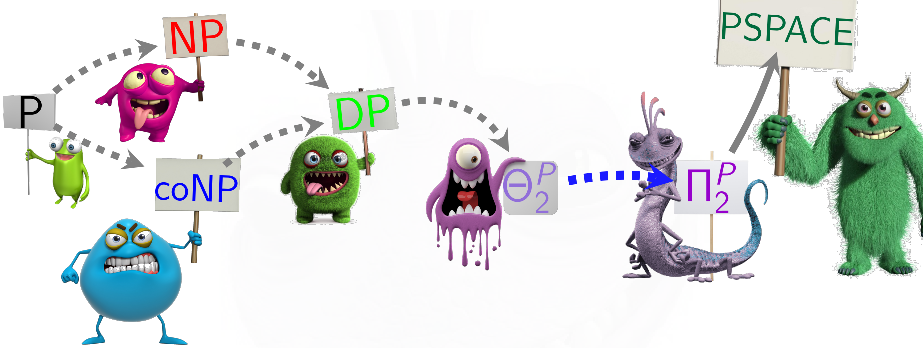


QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard
3. coNP vs NP-hard
4. NP $\cup$ coNP vs DP-hard

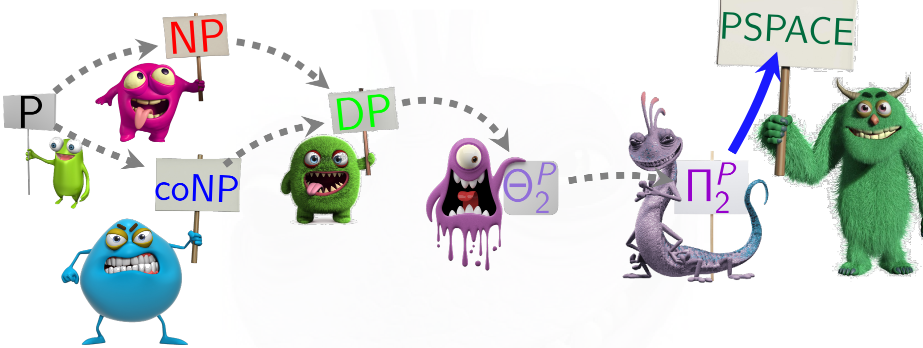
QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard
3. coNP vs NP-hard
4. NP $\cup$ coNP vs DP-hard
5. DP vs Θ_2^P-hard



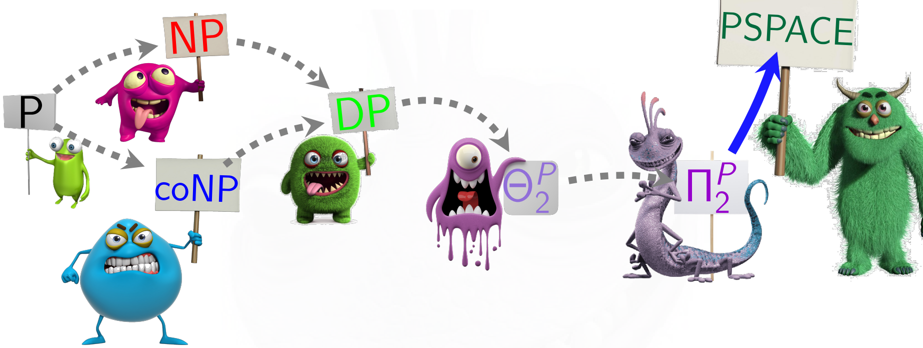
QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard
3. coNP vs NP-hard
4. NP ∪ coNP vs DP-hard
5. DP vs Θ_2^P-hard
6. Θ_2^P vs Π_2^P-hard



QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard
3. coNP vs NP-hard
4. NP $\cup$ coNP vs DP-hard
5. DP vs Θ_2^P-hard
6. Θ_2^P vs Π_2^P-hard
7. Π_2^P vs PSpace-hard



QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard
3. coNP vs NP-hard
4. NP $\cup$ coNP vs DP-hard
5. DP vs Θ_2^P-hard
6. Θ_2^P vs Π_2^P-hard
7. Π_2^P vs PSpace-hard (proved for Γ containing $\{x = a \mid a \in A\}$)

Thank you for your attention

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