Quantified Constraint Satisfaction Problem: towards the classification of complexity

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Quantified Equality Constraints

$(\mathbb{N}; =)$

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$(\mathsf{N}; =)$

$8x_1 9x_2 8x_3 9x_4 (x_1 = x_2 \,\hat{}\, x_3 = x_4)$,

Quantified Equality Constraints

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QCSP(N; $x = y$)

Given a sentence $8x_1 9x_2 ::: 8x_n \;\; {}_1 9x_n \, (x_{i_1} = x_{j_1} \wedge \qquad \wedge x_{i_s} = x_{j_s})$.
Decide whether it holds.

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QCSP$(\mathbb{N}; R)$

Given a sentence $8x_1 9x_2 ::: 8x_{n\ \ 1} 9x_n (R(:::) \wedge \quad \wedge R(:::))$.
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QCSP$(\mathbb{N}; R)$

Given a sentence $\forall x_1 \exists x_2 \dots \forall x_{n-1} \exists x_n \, (R(\dots) \wedge \cdots \wedge R(\dots))$.
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- QCSP$(\mathbb{N}; x = y)$ is solvable in polynomial time.
- QCSP$(\mathbb{N}; x = y \vee z = t)$ is NP-complete [Bodirsky, Chen 2007].

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What is the complexity of QCSP$(\mathbb{N}; x = y \rightarrow y = z)$?

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A concrete question

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QCSP$(\mathbb{N}; R)$

Given a sentence $8x_1 9x_2 : : : 8x_{n \;\; 1} 9x_n \, (R(: : :) \wedge \quad \wedge R(: : :))$.
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Accessible to anyone

Easy to Formulate

Quantified Equality Constraints

What is the complexity of QCSP(N; $x = y$! $y = z$)?

Quantified Equality Constraints

What is the complexity of QCSP$(\mathrm{N}; x = y \; ! \; y = z)$?

- QCSP$(\mathrm{N}; x = y \; ! \; y = z)$ is coNP-hard [Bodirsky, Chen, 2010].

Quantified Equality Constraints

What is the complexity of $\mathrm{QCSP}(\mathbb{N}; x = y \; ! \; y = z)$?

- $\mathrm{QCSP}(\mathbb{N}; x = y \; ! \; y = z)$ is coNP-hard [Bodirsky, Chen, 2010].

Lemma [Zhuk, Martin, 2021]

$\mathrm{QCSP}(\mathbb{N}; x = y \; ! \; y = z)$ is PSpace-hard.

Quantified Equality Constraints

What is the complexity of QCSP(N; $x = y \ ! \ y = z$)?

- QCSP(N; $x = y \ ! \ y = z$) is coNP-hard [Bodirsky, Chen, 2010].

Lemma [Zhuk, Martin, 2021]

QCSP(N; $x = y \ ! \ y = z$) is PSpace-hard.

Theorem [Zhuk, Martin, Bodirsky, Chen, 2021]

Suppose relations $R_1; : : : ; R_s$ are definable by some Boolean combination of atoms of the form $(x = y)$. Then QCSP(N; $R_1; : : : ; R_s$) is either tractable, NP-complete, or PSpace-complete.

Quantified Constraint Satisfaction Problem

A is a finite set,
Γ is a set of relations on A (a constraint language)

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QCSP(Γ):

Given a sentence $9y_1 8x_1 ::: 9y_t 8x_t (R_1(:::) \wedge \quad \wedge R_s(:::))$, where $R_1; :::; R_s$ 2 Γ.
Decide whether it holds.

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Examples:
$A = f0; 1; 2g; \Gamma = fx \neq yg$.

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Given a sentence $\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$, where $R_1, \ldots, R_s \in \Gamma$.
Decide whether it holds.

Examples:
$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$. QCSP instances:

$\forall x \exists y_1 \exists y_2 (x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2)$,

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Examples:
$A = f0; 1; 2g; \Gamma = fx \; 6= yg$. QCSP instances:

$8x 9y_1 9y_2 (x \; 6= y_1 \wedge x \; 6= y_2 \wedge y_1 \; 6= y_2)$, true

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$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (x_1 \neq y_1 \wedge y_1 \neq y_2 \wedge y_2 \neq x_2)$,

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$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (x_1 \neq y_1 \wedge y_1 \neq y_2 \wedge y_2 \neq x_2)$, true

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$8x 9y_1 9y_2 (x \; 6\!\!= y_1 \wedge x \; 6\!\!= y_2 \wedge y_1 \; 6\!\!= y_2)$, true

$8x_1 8x_2 8x_3 9y (x_1 \; 6\!\!= y \wedge x_2 \; 6\!\!= y \wedge x_3 \; 6\!\!= y)$, false

$8x_1 9y_1 8x_2 9y_2 (x_1 \; 6\!\!= y_1 \wedge y_1 \; 6\!\!= y_2 \wedge y_2 \; 6\!\!= x_2)$, true

Main Question

What is the complexity of QCSP(Γ) for different Γ?

Σ	dual-Σ	Classification	Complexity Classes

Quantified Constraint Satisfaction Problem:

Given a sentence $\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$, where $R_1, \ldots, R_s \in \Gamma$.

Decide whether it holds.

Σ	dual-Σ	Classification	Complexity Classes
$f9; 8; \wedge g$			

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Given a sentence $9y_1 8x_1 ::: 9y_t 8x_t (R_1(:::) \wedge \quad \wedge R_s(:::))$,
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Σ	dual-Σ	Classification	Complexity Classes
$f9;8;$^g	$f9;8;\_g$		

Quantified Constraint Satisfaction Problem:

Given a sentence $9y_1 8x_1 ::: 9y_t 8x_t(R_1(:::)$ ^ ^ $R_s(:::))$,
where $R_1; :::; R_s$ 2 Γ.

Decide whether it holds.

Σ	dual-Σ	Classification	Complexity Classes
$f9;8;\wedge g$	$f9;8;\vee g$	??????????	??????????

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Decide whether it holds.

Σ	dual-Σ	Classification	Complexity Classes
f9; 8; ^g	*f9; 8; _g*	??????????	??????????
f9; _g	*f8; ^g*	Trivial	L

Given a sentence $9y_1 ::: 9y_t(R_1(:::) \_ \quad \_ R_s(:::))$,
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Σ	dual-Σ	Classification	Complexity Classes
f9; 8; ^g	f9; 8; _g	??????????	??????????
f9; _g	f8; ^g	Trivial	L
f9; ^g	f8; _g	CSP Dichotomy	P, NP-complete

Constraint Satisfaction Problem:
Given a sentence $9y_1 ::: 9y_t(R_1(:::) \hat{} \quad \hat{} R_s(:::))$,
where $R_1; ::: ; R_s$ 2 Γ.
Decide whether it holds.

Σ	dual-Σ	Classification	Complexity Classes
$f9; 8; \wedge g$	$f9; 8; \_g$	??????????	??????????
$f9; \_g$	$f8; \wedge g$	Trivial	L
$f9; \wedge g$	$f8; \_g$	CSP Dichotomy	P, NP-complete
$f9; \wedge; \_g$	$f8; \wedge; \_g$	Trivial iff the core has one element	L NP-complete

Given a sentence $9y_1 ::: 9y_t((R_1(:::) \_ R_2(:::)) \wedge R_3(:::))$, where $R_1; :::; R_3$ 2 Γ.

Decide whether it holds.

Σ	dual-Σ	Classification	Complexity Classes
f9; 8; ^g	*f9; 8; _g*	??????????	??????????
f9; _g	*f8; ^g*	Trivial	L
f9; ^g	*f8; _g*	CSP Dichotomy	P, NP-complete
f9; ^; _g	*f8; ^; _g*	Trivial iff the core has one element	L NP-complete
f9; 8; ^; _g		Positive equality free tetrachotomy	P, NP-complete co-NP-complete PSPACE-complete

Given a sentence $9y_1 8x_1 ::: 9y_t 8x_t((R_1(:::) \_ R_2(:::)) \,\hat{}\, R_3(:::))$, where $R_1; :::; R_3$ 2 Γ.
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f9; 8; ^g	*f9; 8; _g*	??????????	??????????
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Given a sentence
$9y_1 8x_1 ::: 9y_t 8x_t((:R_1(:::)\ \_\ R_2(:::))\ \hat{}\ :R_3(:::))$,
where $R_1; ::: ; R_3\ 2\ \Gamma$.
Decide whether it holds.

Σ	dual-Σ	Classification	Complexity Classes
f9; 8; ^g	f9; 8; _g	??????????	??????????
f9; _g	f8; ^g	Trivial	L
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f9; 8; ^; _; :g		Trivial iff Γ is trivial	L PSPACE-complete

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Given a sentence $9y_1 8x_1 : : : 9y_t 8x_t (R_1(: : :) \,\hat{}\, \quad \hat{}\, R_s(: : :))$,
where $R_1; : : : ; R_s\ 2\ \Gamma$.
Decide whether it holds.

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CSP(Γ):

Given a formula $(R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$,
where $R_1, \ldots, R_s \in \Gamma$.
Decide whether the formula is satisfiable.

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(equivalently, f is a polymorphism of R, shortly f 2 Pol(R))

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if for all $\begin{pmatrix} a_1^1 \\ \vdots \\ a_1^s \end{pmatrix}, \dots, \begin{pmatrix} a_n^1 \\ \vdots \\ a_n^s \end{pmatrix} \in R;$

$$f \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^s & \dots & a_n^s \end{pmatrix} = \begin{pmatrix} f(a_1^1, \dots, a_n^1) \\ \vdots \\ f(a_1^s, \dots, a_n^s) \end{pmatrix} \in R$$

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if for all $\begin{pmatrix} a_1^1 \\ \vdots \\ a_1^s \end{pmatrix}, \dots, \begin{pmatrix} a_n^1 \\ \vdots \\ a_n^s \end{pmatrix} \in R$;

$$f \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^s & \dots & a_n^s \end{pmatrix} = \begin{pmatrix} f(a_1^1, \dots, a_n^1) \\ \vdots \\ f(a_1^s, \dots, a_n^s) \end{pmatrix} \in R$$

f preserves Γ (equivalently $f \in \mathrm{Pol}(\Gamma)$) if f preserves every $R \in \Gamma$.

Constraint Satisfaction Problem

A is a finite set,
Γ is a set of relations on A (a constraint language)

CSP(Γ):

Given a sentence $9y_1 ::: 9y_t(R_1(:::) \wedge \quad \wedge R_s(:::))$,
where $R_1; :::; R_s \; 2 \; \Gamma$.
Decide whether it holds.

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Theorem [Bulatov, Zhuk, 2017]

- CSP(Γ) is solvable in polynomial time (tractable) if there exists a weak near-unanimity operation preserving Γ,
- CSP(Γ) is NP-complete otherwise.

Weak near-unanimity operation (WNU) is an operation satisfying

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Examples: $x \vee y, x \wedge y, xy \vee xz \vee yz, x + y + z, 0, \min(x, y), \dots$

Few facts about QCSP

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- If Γ contains all relations then QCSP(Γ) is PSPACE-complete.

PSPACE

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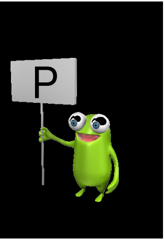
- If Γ consists of linear equations in a finite field then QCSP(Γ) can be solved in polynomial time (tractable).



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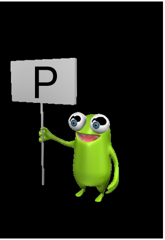

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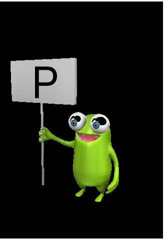

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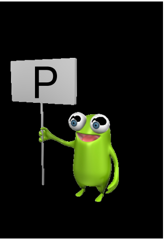
Are there any other complexity classes?



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- Boolean structures. Dichotomy P, Pspace-complete. (Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.)

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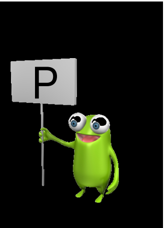

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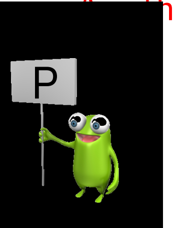

Surjective polymorphisms

Observation

Suppose each relation of Γ_1 is definable from Γ_2 using quantified conjunctive formulas

$$R(x_1, \dots, x_n) = \forall y_1 \exists y_2 \forall y_3 \exists y_4 \dots R_1(\dots) \wedge \cdots \wedge R_s(\dots).$$

Then QCSP(Γ_1) is polynomially reducible to QCSP(Γ_2).

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Theorem (Galois Correspondence, Börner, Bulatov, Chen, Jeavons, and Krokhin, 2003)

Γ_1 is de nable by quanti ed conjunctive formulas over Γ_2 IFF each surjective polymorphism of Γ_2 is a polymorphism of Γ_1.

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QCSP ^{2}( ) :

Given a sentence $\forall x_1 \ldots \forall x_t \exists y_1 \ldots \exists y_q (R_1(\ldots) \wedge \quad \wedge R_s(\ldots))$, where $R_1, \ldots, R_s \in$.

Decide whether it holds.

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- How many tuples is it sufficient to check?

PGP vs EGP

For an algebra $(A; F)$ (a set of operations F on a set A)
$d_F(n)$ is the minimal size of a generating set of A^n.

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Examples

1. $A = \{0;1\}$, $F = \{x \vee y\}$. $d_F(n) = n+1$. It is sufficient to have $(0; \ldots ; 0)$ and $(0; \ldots ; 0; 1; 0; \ldots ; 0)$ for any position of 1 to generate $\{0;1\}^n$.

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Pair $(a_i; a_{i+1})$ with $a_i \neq a_{i+1}$ is a switch in a tuple $(a_1; \ldots; a_n)$.
$(0; 0; 0; 1; 2; 2; 0; 0; 0; 0)$ has 3 switches,
$(3; 3; 3; 4; 3; 3; 3; 3; 3; 3)$ has 2 switches.

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Theorem[Zhuk, 2015]

A finite algebra A has PGP IFF there exists k such that each A^n is generated by all tuples with at most k switches.

From ${}_2$ to NP

QCSP^2( ) :

Given a sentence $\forall x_1 \ldots \forall x_t \exists y_1 \ldots \exists y_q (R_1(\ldots) \wedge \quad \wedge R_s(\ldots))$,
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If $x \vee y$ preserves then it is sufficient to check that
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and $(x_1, \dots, x_{i \; 1}, x_i, x_{i+1}, \dots, x_t) = (0, \dots, 0, 1, 0, \dots, 0)$ for $\forall i$.

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Observation

If Pol( ) has PGP, then QCSP^2( ) can be polynomially reduced to CSP($\cup \{ x = a \mid a \in A\}$).

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Proof:

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Proof: the instance is equivalent to the CSP instance

$$\bigwedge_{\substack{(a_1,\dots,a_t) \text{ with} \\ \text{at most } k \text{ switches}}} (R_1(\dots) \wedge \quad \wedge R_s(\dots) \wedge (x_1 = a_1) \wedge \quad \wedge (x_t = a_t))$$

From PSpace to NP

From PSpace to NP

9y8x

From PSpace to NP

$$\exists y \forall x$$

$$m$$

$$\forall x^1 \forall x^2 \dots \forall x^{|A|} \exists y \quad _1 \wedge \quad _2 \wedge \quad \wedge \quad _{|A|}$$

- $_i$ is obtained from by renaming x by x^i

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From PSpace to NP

$$9y\,8x$$

$$m$$

$$8x^1 8x^2 ::: 8x^{jAj} 9y \quad {}_1 \wedge \quad {}_2 \wedge \quad \wedge \quad {}_{jAj}$$

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$$9y_1 9y_2^1 ::: 9y_2^{jAj} ::: \; 9y_t^1 ::: 9y_t^{jAj^{t-1}} \quad {}_1 \wedge \quad {}_2 \wedge \quad \wedge \quad {}_q$$

From PSpace to NP

$$\exists y \forall x$$
$$\Updownarrow$$
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$$\exists y_1 \exists y_2^1 \ldots \exists y_2^{|A|} \ldots \; \exists y_t^1 \ldots \exists y_t^{|A|^{t \; 1}} \quad _1 \wedge \quad _2 \wedge \quad \wedge \quad _q$$

- For the PGP case it is su cient to check tuples with at most k switches

From PSpace to NP

$$9y\,8x$$

$$m$$

$$8x^{1}8x^{2} : : : 8x^{jAj}9y \quad _1 \wedge \quad _2 \wedge \quad \wedge \quad _{jAj}$$

- $_i$ is obtained from by renaming x by x^i

$$9y_1 8x_1 : : : 9y_t\, 8x_t$$

$$m$$

$$1\ 1\ 1 : : : 1\ 1 \quad 1\ 1\ 2 : : : 2 \quad : : : \quad 0\ 0\ 0\ 0 : : : 0$$

$$8x_1^1 : : : 8x_1^{jAj}\ 8x_2^1 : : : 8x_2^{jAj^2}\ : : : 8x_t^1 : : : 8x_t^{jAj^t}$$

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- ${}_i$ is obtained from by renaming x by x^i

$$9y_1 8x_1 :::9y_t 8x_t$$
$$m$$
$$1\ 1\ 1:::1\ 1 \quad 1\ 1\ 2:::2 \quad ::: \quad 0\ 0\ 0\ 0:::0$$
$$8x_1^1 :::8x_1^{jAj}\ 8x_2^1 :::8x_2^{jAj^2} :::8x_t^1 :::8x_t^{jAj^t}$$
$$9y_1 9y_2^1 :::9y_2^{jAj} ::: \ 9y_t^1 :::9y_t^{jAj^{t}\ 1} \quad {}_1 \wedge \quad {}_2 \wedge \quad \wedge \quad {}_q$$

- For the PGP case it is su cient to check tuples with at most k switches
- We keep variables with the switches

From PSpace to NP

$$\exists y \forall x$$

$$\Leftrightarrow$$

$$\forall x^1 \forall x^2 \dots \forall x^{|A|} \exists y \quad _1 \wedge \quad _2 \wedge \quad \wedge \quad _{|A|}$$

- $_i$ is obtained from by renaming x by x^i

$$\exists y_1 \forall x_1 \dots \exists y_t \forall x_t$$

$$\Leftrightarrow$$

$$1\ 1\ 1 \dots 1\ 1 \quad 1\ 1\ 2 \dots 2 \quad \dots \quad 0\ 0\ 0\ 0 \dots 0$$

$$\forall x_1^1 \dots \forall x_1^{|A|} \; \forall x_2^1 \dots \forall x_2^{|A|^2} \dots \forall x_t^1 \dots \forall x_t^{|A|^t}$$

$$\exists y_1 \exists y_2^1 \dots \exists y_2^{|A|} \dots \; \exists y_t^1 \dots \exists y_t^{|A|^{t-1}} \quad _1 \wedge \quad _2 \wedge \quad \wedge \quad _q$$

- For the PGP case it is su cient to check tuples with at most k switches
- We keep variables with the switches
- We assign $x_1^1 = \quad = x_1^{|A|} = 1; \dots; x_t^1 = \quad = x_t^{|A|^t} = 0$

From PSpace to NP

Theorem

Suppose Pol( ) has PGP. Then QCSP( ) is polynomially reducible to CSP($\cup \{ \ x = a \mid a \in A\}$).

From PSpace to NP

Theorem*

Suppose Pol( ) has PGP. Then QCSP( ) is polynomially reducible to CSP([f x = a j a 2 Ag).

* For   containing all constants relations this was shown earlier by Chen, Martin, Carvalho, and Madelaine.

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Suppose Pol( ) has PGP. Then QCSP( ) is in NP.

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Corollary 1

Suppose Pol( ) has PGP. Then QCSP( ) is in NP.

Corollary 2

Suppose Pol( ) has PGP. Then QCSP( ) is either tractable, or NP-complete.

Chen Conjecture

Suppose contains $\{x = a \mid a \in A\}$. Then QCSP( )

Chen Conjecture

Suppose contains $\{x = a \mid a \in A\}$. Then QCSP( )

- is in P, if Pol( ) has PGP and WNU

P

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Suppose contains $\{x = a \mid a \in A\}$. Then QCSP( )

- is in P, if Pol( ) has PGP and WNU
- is NP-complete, if Pol( ) has PGP and has no WNU

P

NP

Chen Conjecture (QCSP Trichotomy Conjecture)

Suppose contains $\{x = a \mid a \in A\}$. Then QCSP( )

- is in P, if Pol( ) has PGP and WNU
- is NP-complete, if Pol( ) has PGP and has no WNU
- is PSPACE-complete, if Pol( ) has no PGP

P

PSPACE

NP

Chen Conjecture

Weak Chen Conjecture

If Pol( ) has EGP, then QCSP( ) is coNP-hard.

Chen Conjecture

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Almost a proof of Weak Chen Conjecture

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Almost a proof of Weak Chen Conjecture

1. If Pol( ) has EGP then we can de ne (encode) by a positive primitive formula the compliment to 3-CNF.

Chen Conjecture

Weak Chen Conjecture

If Pol( ) has EGP, then QCSP( ) is coNP-hard.

Almost a proof of Weak Chen Conjecture

1. If Pol( ) has EGP then we can de ne (encode) by a positive primitive formula the compliment to 3-CNF.
2. If this de nition is e ciently computable, then QCSP( ) is coNP-hard.

Chen Conjecture

Weak Chen Conjecture

If Pol( ) has EGP, then QCSP( ) is coNP-hard.

Almost a proof of Weak Chen Conjecture

1. If Pol( ) has EGP then we can de ne (encode) by a positive primitive formula the compliment to 3-CNF.
2. If this de nition is e ciently computable, then QCSP( ) is coNP-hard.

Lemma (Classi cation for the conservative case) [Zhuk, Martin, 2018]

Chen Conjecture holds for containing all unary relations.

QCSP Monsters

P

PSPACE

NP

QCSP Monsters

- there exists   on a 3-element domain such that QCSP( ) is coNP-complete.

P

PSPACE

coNP

NP

QCSP Monsters

- there exists   on a 3-element domain such that QCSP( ) is coNP-complete.
- there exists   on a 4-element domain such that QCSP( ) is DP-complete, where DP = NP^ coNP.

DP

coNP

P

PSPACE

NP

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- there exists on a 3-element domain such that QCSP( ) is coNP-complete.
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- there exists on a 10-element domain such that QCSP( ) is $^{P}_{2}$-complete.

DP

coNP

P

PSPACE

NP

$^{P}_{2}$

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DP

coNP

P

PSPACE

NP

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DP

Are there any other monsters???

coNP

P

PSPACE

NP

$^{P}_{2}$

Classi cation for a 3-element-domain

Classification for a 3-element-domain

Theorem (Classification for a 3-element domain)

Suppose Γ is a constraint language on $\{0,1,2\}$ containing $\{x = a \mid a \in \{0,1,2\}\}$. Then QCSP($\Gamma$) is

- in P, or
- NP-complete, or
- coNP-complete, or
- PSPACE-complete.

Classification for a 3-element-domain

Theorem (Classification for a 3-element domain)

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- in P, or
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PSPACE

NP

coNP

P

Two questions

- What makes QCSP( ) easy?
- What makes QCSP( ) hard?

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What makes QCSP( ) PSpace-hard?

Let A = f + ; ; 0, 1 g, = f R_0; R_1; f + g; f gg .

What makes QCSP( ) PSpace-hard?

Let A = f + ; ; 0, 1 g, = f R_0; R_1; f + g; f gg .

$R_0(y_1; y_2; x)$ = ($y_1; y_2$ 2 f + ; g) ^ (y_1 = y_2 _ x 6= 0)

What makes QCSP( ) PSpace-hard?

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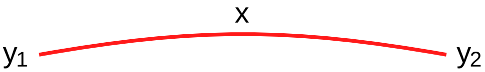

What makes QCSP( ) PSpace-hard?

Let A = f + ; ; 0, 1 g, = f $R_0; R_1$; f + g; f gg .

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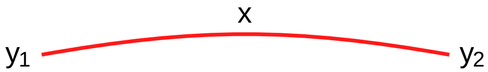

$R_1(y_1; y_2; x)$ = ($y_1; y_2$ 2 f + ; g) ^ (y_1 = y_2 _ x 6= 1)

What makes QCSP( ) PSpace-hard?

Let $A = \{+, \ , 0, 1\}$, $\ = \{R_0, R_1, \{+\}, \{\ \}\}$.

$R_0(y_1, y_2, x) = (\ y_1, y_2 \in \{+, \ \}\ ) \wedge (y_1 = \ y_2 \vee x \neq 0)$

x

y_1  y_2

$R_1(y_1, y_2, x) = (\ y_1, y_2 \in \{+, \ \}\ ) \wedge (y_1 = \ y_2 \vee x \neq 1)$

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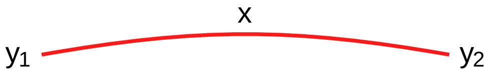

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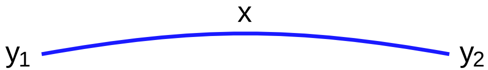
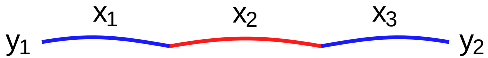


9$u_1$9$u_2$$R_1(y_1; u_1; x_1)$ ^ $R_0(u_1; u_2; x_2)$ ^ $R_1(u_2; y_2; x_3)$



How to prove PSpace-hardness?

Let A = f + ; ; 0, 1 g, = f R_0; R_1; f + g; f gg .

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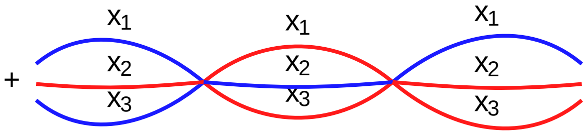

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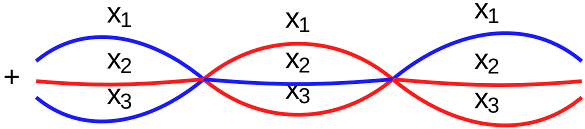

$: ((x_1 \_ \overline{x}_2 \_ x_3)\ \hat{}\ (\overline{x}_1 \_ x_2 \_ \overline{x}_3)\ \hat{}\ (x_1 \_ \overline{x}_2 \_ \overline{x}_3))$

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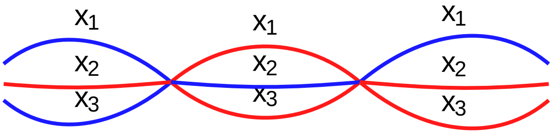

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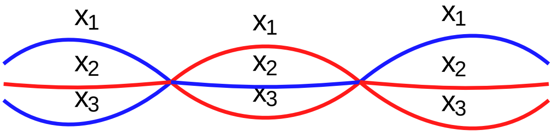

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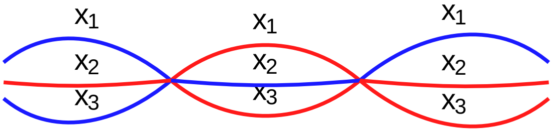

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m

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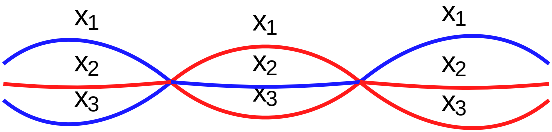

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Claim

QCSP( ) is coNP-hard.

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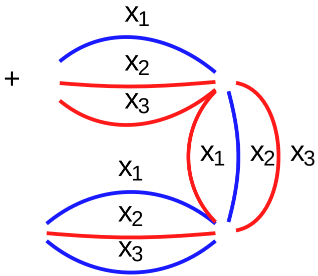

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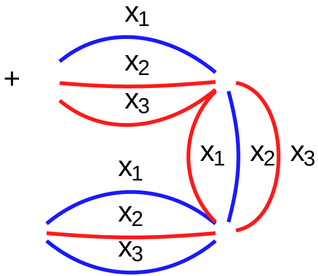

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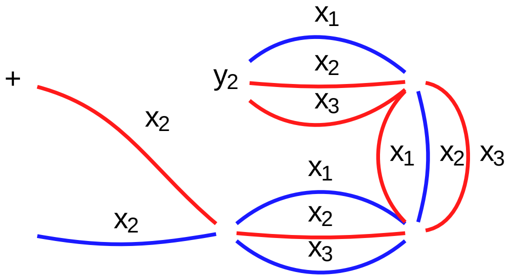

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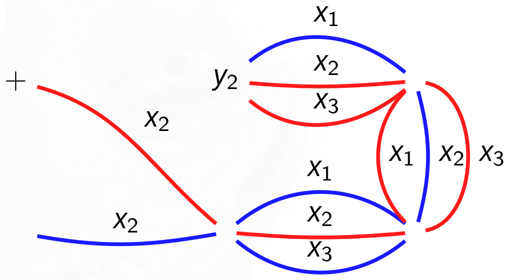

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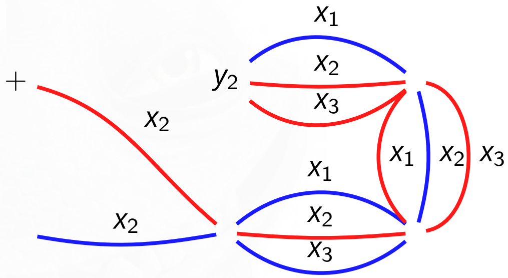

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Claim

QCSP(Γ) is PSpace-hard.

Theorem (Π_2^P vs PSpace)

Suppose Γ *contains* $\{x = a \mid a \in A\}$. *Then* QCSP(Γ)

- *is PSpace-hard if there exists a reflexive relation* $S \subsetneq A^n$ *and a nontrivial equivalence relation* η *on* $D \subseteq A$ *such that* $R(y_1, y_2, x_1, \ldots, x_n) = \eta(y_1, y_2) \vee S(x_1, \ldots, x_n)$ *is definable by a positive primitive formula over* Γ
- *in* Π_2^P *otherwise.*

Theorem (Π_2^P vs PSpace)

Suppose Γ *contains* $f x = a \; j \; a \; 2 \; Ag$. *Then* QCSP($\Gamma$)

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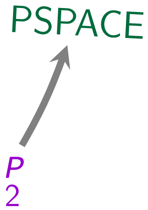

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- *in* Π_2^P *otherwise.*

Lemma

There exists Γ on a 6-element set such that QCSP(Γ) is Π_2^P-complete.


P
NP
coNP
DP
P 2
P 2
PSPACE

QCSP Hepta-chotomy to prove

QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).

QCSP Hepta-chotomy to prove

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2. NP vs coNP-hard

QCSP Hepta-chotomy to prove

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2. NP vs coNP-hard
3. coNP vs NP-hard

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4. NP [coNP vs DP-hard

QCSP Hepta-chotomy to prove

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QCSP Hepta-chotomy to prove

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6. Θ_2^P vs Π_2^P-hard

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3. coNP vs NP-hard
4. NP [coNP vs DP-hard
5. DP vs Θ_2^P-hard
6. Θ_2^P vs Π_2^P-hard
7. Π_2^P vs PSpace-hard

QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard
3. coNP vs NP-hard
4. NP *[* coNP vs DP-hard
5. DP vs Θ_2^P-hard
6. Θ_2^P vs Π_2^P-hard
7. Π_2^P vs PSpace-hard (proved for Γ containing $fx = a\ j\ a\ 2\ Ag$)

Thank you for your attention

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