Amalgamation property for varieties of BL-algebras generated by one chain with finitely many components

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(joint work with Stefano Aguzzoli)

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BL-algebras and BL

Definition 1

A BL-algebra is a commutative, integral, bounded, prelinar, divisible residuated lattice of the form $\mathcal{A} = (A, *, \rightarrow, \wedge, \vee, 0, 1)$. A totally ordered BL-algebra is called BL-chain.

The class of all BL-algebras forms an algebraic variety, called $\mathbb{BL}$. Given a variety $\mathbb{L}$ of BL-algebras, with $Ch(\mathbb{L})$ we denote the class of the chains in $\mathbb{L}$.

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Gödel-algebras (G-algebras) are idempotent BL-algebras, *i.e.* they satisfy $x * x = x$. In every G-chain $x * y = \min\{x, y\}$, and hence for every $n \in \mathbb{N}$ all the G-chains with n elements are isomorphic. With $\mathcal{G}_n$ ($n \geq 3$) we denote the G-chain with lattice reduct $0 < \frac{1}{n-1} < \cdots < \frac{n-1}{n-1}$.

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For $k \geq 1$, the variety $\mathbb{P}_k$ is such that each of its chains is isomorphic to an ordinal sum $\mathcal{A}_0 \oplus \cdots \oplus \mathcal{A}_k$, where $\mathcal{A}_0$ as at most two elements, and for $i > 0$, $\mathcal{A}_i$ is a totally ordered cancellative hoop (possibly trivial). For $k \geq 1$ we have $\mathbb{P}_k = \mathbf{V}(\mathcal{P}_k)$.

Definition 3

We say that a class K of BL-algebras has the *amalgamation property* (AP) if for every 5-tuple (called V-formation) $(\mathcal{A}, \mathcal{B}, \mathcal{C}, i, j)$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$ and $\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{B}$, $\mathcal{A} \overset{j}{\hookrightarrow} \mathcal{C}$, there is a triple (called amalgam) $(\mathcal{D}, h, k)$, with $\mathcal{D} \in K$, $\mathcal{B} \overset{h}{\hookrightarrow} \mathcal{D}$, $\mathcal{C} \overset{k}{\hookrightarrow} \mathcal{D}$, such that $h \circ i = k \circ j$.

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For the varieties of BL-algebras a sufficient condition for the AP is the following.

Theorem 4 ([Mon06], [MMT14])

Let $\mathbb{L}$ be a non-trivial variety of BL-algebras. If $Ch(\mathbb{L})$ enjoys the AP then the same holds for $\mathbb{L}$.

The AP for varieties of MV-algebras has already been classified.

Theorem 5 ([DL00])

A variety $\mathbb{L}$ of MV-algebras has the AP if and only if it is single-chain generated, i.e. $\mathbb{L} = \mathbf{V}(\mathcal{A})$, *for some* $\mathcal{A} \in Ch(\mathcal{A})$.

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$Ch(\mathbb{BL})$ *has the AP, and hence the variety of BL-algebras has the AP.*

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Differently from MV-algebras, it is not true that a variety of BL-algebras has the AP iff it is single-chain generated.

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For every $n \geq 4$ the variety $\mathbb{G}_n$ generated by the n-element Gödel-chain, does not have the AP.

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In this talk we will classify the AP for varieties of BL-algebras which are generated by one chain with finitely-many components.

AP for (some) varieties of BL-algebras: negative cases

Lemma 8

Let $\mathbb{L}$ be a variety of BL-algebras. Then the following are equivalent.

1. *$\mathbb{L}$ contains neither $\mathcal{G}_4$ nor $\mathcal{P}_2$.*
2. *Every chain $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$ in $\mathbb{L}$ is such that $|I| \leq 3$, there is at most one $i \in I \setminus \{0\}$ such that $\mathcal{A}_i$ is infinite, and there is at most one $j \in I \setminus \{0\}$ such that $\mathcal{A}_j$ is bounded.*

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Theorem 9

Let $\mathbb{L}$ be a variety of BL-algebras such that every chain has finitely many components. If $\mathbb{L}$ contains $\mathcal{G}_4$ or $\mathcal{P}_2$, then $\mathbb{L}$ does not have the AP.

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Proposition 1

Let $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$ be a BL-chain. Suppose that:

- *$|I| \geq 2$, i.e., $\mathcal{A}$ has at least two components.*
- *$\mathcal{A}_0$ is an MV-chain with infinite rank such that $\mathcal{L}_k \not\hookrightarrow \mathcal{A}_0$, for some $k \geq 3$ or $\mathcal{A}_0$ is an infinite MV-chain with rank $k \geq 3$, and $\mathcal{L}_k \not\hookrightarrow \mathcal{A}_0$.*

Then $\mathbf{V}(\mathcal{A})$ does not have the AP.

Theorem 10

Let $\mathbb{L}$ be a variety of BL-algebras generated by one chain with finitely many components. Then the following are equivalent:

(i) $\mathbb{L}$ *has the AP.*

(ii) *Every BL-chain* $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$ *such that* $\mathbf{V}(\mathcal{A}) = \mathbb{L}$ *satisfies the following conditions.*

- $|I| \leq 3$.
- *There is at most one* $i \in I \setminus \{0\}$ *such that* $\mathcal{A}_i$ *is infinite, and there is at most one* $j \in I \setminus \{0\}$ *such that* $\mathcal{A}_j$ *is bounded.*
- *If* $|I| \geq 2$ *then the following ones hold.*
  - *If* $\mathcal{A}_0$ *has infinite rank, then* $\mathcal{L}_k \hookrightarrow \mathcal{A}_0$, *for every* $k \geq 2$.
  - *If* $\mathcal{A}_0$ *is infinite and* $\text{rank}(\mathcal{A}_0) = k$, *then* $\mathcal{L}_k \hookrightarrow \mathcal{A}_0$.

Main result: proof sketch

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Assume now that every generic chain satisfies the theorem's condition, and let $\mathcal{A}$ be one of these chains. We prove that $\mathbf{V}(\mathcal{A}) = \mathbb{L}$ has the AP by showing the AP for $Ch(\mathcal{A})$.

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If $\#\mathcal{A} = 1$, then $\mathcal{A}$ is an MV-chain and by already known results we have that $\mathbb{L}$ has the AP.

Assume now that $\#\mathcal{A} = 2$ or $\#\mathcal{A} = 3$. It can be shown that:

- There exists a BL-chain $\mathcal{A}^s \in \mathbb{L}$ such that $Ch(\mathbf{V}(\mathcal{A})) = \mathbf{ISP}_u(\mathcal{A}^s)$.
- For every totally ordered Wajsberg hoop $\mathcal{H}$, $Ch(\mathcal{H})$ has the AP.

There are many cases to check, but using these results (and others) it can be shown that $Ch(\mathcal{A})$ has the AP, by constructing an amalgam for every possible V-formation of chains in $\mathbb{L}$.

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A first step to solve the problem is the following.

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Open problems

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The requirement that S is finite is non-trivial, since

Theorem 13

Every variety of BL-algebras is generated by some set of BL-chains with finitely many components.

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APPENDIX

Axiomatization of BL

The basic connective are $\{\&, \to, \bot\}$ (formulas built inductively: a theory is a set of formulas). Useful derived connectives are the following ones:

(negation) $$\neg\varphi \stackrel{\text{def}}{=} \varphi \to \bot$$

(conjunction) $$\varphi \wedge \psi \stackrel{\text{def}}{=} \varphi\&(\varphi \to \psi)$$

(disjunction) $$\varphi \vee \psi \stackrel{\text{def}}{=} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

(top) $$\top \stackrel{\text{def}}{=} \neg\bot$$

MTL can be axiomatized by using these axioms and modus ponens: $\frac{\varphi \quad \varphi\to\psi}{\psi}$.

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A2) $(\varphi\&\psi) \to \varphi$

(A3) $(\varphi\&\psi) \to (\psi\&\varphi)$

(A4) $(\varphi\&(\varphi \to \psi)) \to (\psi\&(\psi \to \varphi))$

(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi\&\psi) \to \chi)$

(A5b) $((\varphi\&\psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

(A7) $\bot \to \varphi$

back

Definition 14 ([Fer92, BF00])

A *hoop* is a structure $\mathcal{A} = \langle A, *, \to, 1\rangle$ such that $\langle A, *, 1\rangle$ is a commutative monoid, and $\to$ is a binary operation such that

$$x \to x = 1, \qquad x \to (y \to z) = (x * y) \to z \qquad \text{and} \qquad x * (x \to y) = y * (y \to x).$$

Definition 15

A *bounded* hoop is a hoop whose language is expanded with a constant 0 such that $0 \leq x$, for every element x; conversely, an *unbounded* hoop is a hoop without minimum.

Proposition 2 ([Fer92, BF00, AFM07])

- *A hoop is Wajsberg iff it satisfies the equation* $(x \to y) \to y = (y \to x) \to x$.
- *A hoop is cancellative iff it satisfies the equation* $x = y \to (x * y)$.
- *Totally ordered cancellative hoops coincide with unbounded totally ordered Wajsberg hoops, whereas bounded Wajsberg hoops coincide with MV-algebras.*

back

Ordinal Sums

- Let $\langle I, \leq \rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a totally ordered Wajsberg hoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that $\mathcal{A}_0$ is bounded.
- Then $\bigoplus_{i \in I} \mathcal{A}_i$ (the *ordinal sum* of the family $(\mathcal{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom is the minimum of $\mathcal{A}_0$, whose top is 1, and whose operations are

A_j

A_i

$$x \to y = \begin{cases} x \to^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

$$x * y = \begin{cases} x *^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\},\ y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\}, x \in A_j) \end{cases}$$

- As a consequence, if $x \in A_i \setminus \{1\}$, $y \in A_j$ and $i < j$ then $x < y$.

back

BL-algebras

A BL-algebra is an algebra $\langle A, *, \to, \wedge, \vee, 0, 1\rangle$. such that:

1. $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with minimum 0 and maximum 1.
2. $\langle A, *, 1\rangle$ is a commutative monoid.
3. $\langle *, \to\rangle$ forms a *residuated pair*: $z * x \leq y$ iff $z \leq x \to y$ for all $x, y, z \in A$. In particular, it holds that $x \to y = \max\{z \in A : z * x \leq y\}$.
4. The following equations hold.

(Prelinearity) $\quad (x \to y) \vee (y \to x) = 1.$

(Divisibility) $\quad (x \wedge y) = x * (x \to y).$

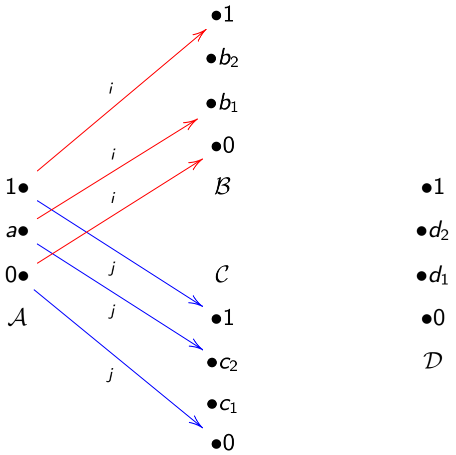
A totally ordered BL-algebra is called *BL-chain*.

- The class of BL-algebras forms a variety, called $\mathbb{BL}$. The logic corresponding to BL-algebras is called BL.
- An axiomatic extension of BL is a logic obtained by adding other axioms to it.
- Every axiomatic extension of BL is algebraizable in the sense of [BP89], and hence every subvariety of $\mathbb{BL}$ induces a logic.

back

The failure of the AP for $Ch(\mathbb{G}_4)$

Pick the V-formation $(\mathcal{A}, \mathcal{B}, \mathcal{C}, i, j)$, with $\mathcal{A}, \mathcal{B}, \mathcal{C} \in Ch(\mathbb{G}_4)$, defined as in the picture.



Since every chain in $\mathbb{G}_4$ has at most 4 elements, then there is no amalgam in $Ch(\mathbb{G}_4)$ for this V-formation, as if $\mathcal{B}, \mathcal{C}$ can be embedded into some $\mathcal{D} \in Ch(\mathbb{G}_4)$, then $\mathcal{B} \simeq \mathcal{C} \simeq \mathcal{D}$.

back

Consider the set $C_\infty = \{x \in \mathbb{Z} : x \leq 0\}$. The hoop $\mathcal{C}_\infty = (C_\infty, *, \Rightarrow, 1)$ is defined as follows, for $x, y \in C_\infty$:

- $1^{\mathcal{C}_\infty} = 0$,
- $x *^{\mathcal{C}_\infty} y = x + y$,
- $x \Rightarrow^{\mathcal{C}_\infty} y = \begin{cases} 0 & \text{if } x \leq^{\mathbb{Z}} y, \\ y - x & \text{otherwise.} \end{cases}$

A direct inspection shows that $\mathcal{C}_\infty$ is a cancellative hoop, and it is known that $\mathbf{V}(\mathcal{C}_\infty) = \mathbb{CH}$.

For $k \geq 1$, we define $\mathcal{P}_k$ as $\mathbf{2} \oplus \underbrace{\mathcal{C}_\infty \oplus \cdots \oplus \mathcal{C}_\infty}_{k \text{ times}}$. We have that $\mathbf{V}(\mathcal{P}_k) = \mathbb{P}_k$.

back