On the Representation Number of Bipartite Graphs

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Outline

Word-representable graph

Definition

A simple graph $G = (V, E)$ is said to be a *word-representable graph* if it can be presented by a word over V. Any two vertices a and b are adjacent if and only if $ababab a\ldots$ or $bababab\ldots$ appear in the word w.

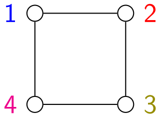


Figure 1 : Cycle C_4.

$w = 413423124$
represents the graph C_4.

- 41414 represents 1 and 4 are adjacent.
- 1331 represents that 1 and 3 are non-adjacent vertices.

k-word-representable graph

- If every letters in w appear exactly k times then w is said to be k-uniform.

Definition

A graph G is said to be a *k-word-representable graph* if there exist a k-uniform word that represents it.

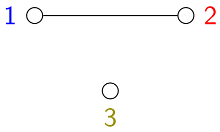


Figure 2 : A Graph G.

- 123312 represents G.
- G is a 2-word-representable graph.

Proposition (Kitaev and Pyatkin (2008))

A k-word-representable graph G is also $(k+1)$-word-representable graph. In particular, each word-representable graph has infinitely many word-representants.

Representation number

Definition

The minimal k such that a graph G is a k-word-representable graph is called the *representation number of G* and is denoted by $\mathcal{R}(G)$.

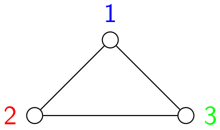


Figure 3 : Graph K_3.

- 123 represents K_3.
- Any complete graph, K_n, on n vertices can be represented by a permutation. So, they have representation number 1.

Permutationally representable graph

Definition

A graph $G = (V, E)$ is said to be a *permtationally representable graph* if it can be represented by a word of the form $w = p_1 p_2 ... p_k$, where p_i is a permutation of V, for all i where $1 \leq i \leq k$. We say G is *permutationally k-representable*.

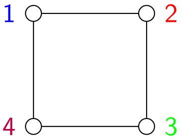


Figure 4 : Cycle C_4.

- $\overbrace{1324}\underbrace{3142}$ represents the graph C_4.
- C_4 is permutationally 2-representable.

Permutation representation number

Definition

The minimal k such that a graph G is a permutationally k-representable graph is called the *permutation representation number of* G, and it is denoted by $\mathcal{R}^p(G)$.

Remark

- *For any permutationally representable graph* G, $\mathcal{R}(G) \leq \mathcal{R}^p(G)$.
- *For all* $n \geq 1$, $\mathcal{R}^p(K_n) = 1$.

Example

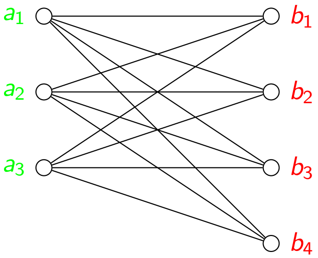


Figure 5 : $K_{3,4}$

- $\overbrace{a_1 a_2 a_3 b_1 b_2 b_3 b_4}\underbrace{a_3 a_2 a_1 b_4 b_3 b_2 b_1}$ represents $K_{3,4}$.
- $\mathcal{R}^p(K_{3,4}) = 2$

Remark

- *Any complete bipartite graph $K_{m,n}$, $\mathcal{R}^p(K_{m,n}) = 2$.*

Semi-transitive Orientation

Definition

A graph $G = (V, E)$ is said to be *semi-transitive* if it admits a semi-transitive orientation, i.e., G admits an acyclic orientation such that for any directed path $v_1 \to v_2 \to \cdots \to v_n$ $\forall v_i, \in V$ where $1 \leq i \leq n$, either

- there is no edge from v_1 to v_n, or
- if there is an edge from v_1 to v_n then there are edges from v_i to v_j, $\forall\ 1 \leq i < j \leq n$.

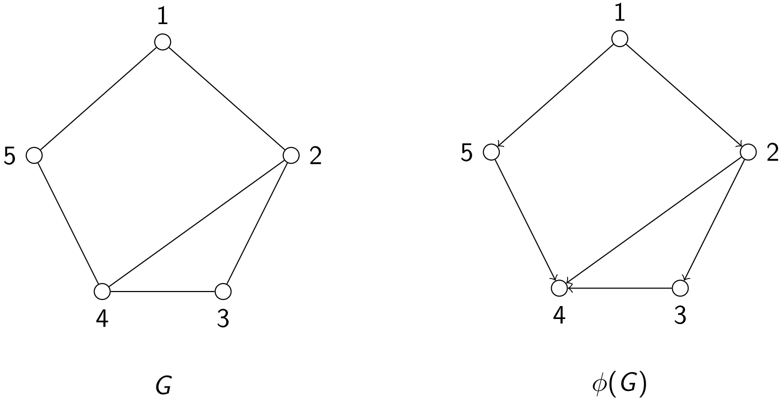


Figure 6 : Semi-transitive orientation of G.

Theorem (Halldórsson et al. (2016))

A graph is word-representable if and only if it admits a semi-transitive orientation.

Theorem (Halldórsson et al. (2016))

Each non-complete word-representable graph G on n vertices is $2(n - \mathcal{K}(G))$-word-representable, where $\mathcal{K}(G)$ is the size of the maximum clique in G.

Corollary

If G is non-complete word-representable on n vertices, then $\mathcal{R}(G) \leq 2(n - \mathcal{K}(G))$.

Computation of word-representable graphs

- The recognition problem for word-representable graph is decidable.
- In Halldórsson et al. (2016), it is established that the recognition problem for word-representable graph is NP-complete.

Classes of graphs which are word-representable

- All graphs on atmost 5-vertices are word-representable.
- Complete graphs.
- Comparability graphs.
- There are classes of graphs that contain both word-representable and non-word-representable graphs: 4-colourable graphs, perfect graphs, etc.

Comparability graphs

Definition

A graph $G = (V, E)$ is a comparability graph if it admits a transitive orientation, i.e, an orientation in which if $\overrightarrow{xy}$ and $\overrightarrow{yz}$ are directed edges, then $\overrightarrow{xz}$ is a directed edge, for all $x, y, z \in V$.

Figure 7 : Transitive orientation of H.

Theorem (Kitaev and Seif (2008))

A graph is permutationally representable if and only if it is a comparability graph.

- Every comparability graph corresponds to a partially ordered set (poset).
- In Yannakakis (1982), the problem of finding the dimension of a poset is NP-hard.

Result (Kitaev and Lozin (2015))

A comparability graph H is permutationally k-representable if and only if the poset induced by this graph has dimension at most k.

Crown graph

Definition

Crown graphs are graphs which are obtained from a complete bipartite graph by removing a perfect matching.

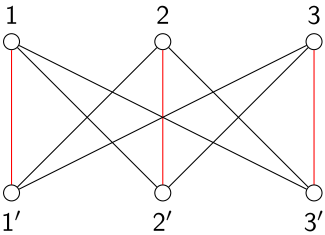


Figure 8 : $K_{3,3}$

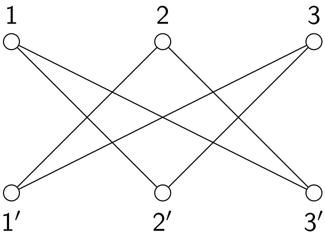


Figure 9 : $H_{3,3}$

Theorem (Halldorsson et al. (2011))

The permutation representation number of a crown graph $H_{n,n}$ is n.

Theorem (Glen et al. (2018))

The representation number of a crown graph $H_{n,n}$ is $\lceil \frac{n}{2} \rceil$.

Conjecture (Glen et al. (2018))

Every bipartite graph on n vertices has representation number at most $\lceil \frac{n}{4} \rceil$.

Main Result

- Every bipartite graph is a comparability graph.
- The class of bipartite graph is precisely the class of comparability graphs that are isomorphic to the Hasse diagram of the corresponding posets.
- We devise an algorithmic procedure that works in polynomial time to construct a word representing permutationally a given bipartite graph.

Relabeling Algorithm

1. Given a bipartite graph $G = (V, E)$ where $V = V_1 \cup V_2$, consider the set, say V_1, with minimum number of vertices.
2. Check whether the graph is a complete bipartite graph. If yes, exit.
3. Else, from the set V_1 choose a vertex with atleast one non adjacent vertex and relabel it first.
4. Relabel the rest of vertices in V_1 and with respect to the relabeling done in V_1, we relabel the vertices in V_2 accordingly.
5. We then create permutations(linear orders) with respect to the vertices in V_1.
6. Concatenate all the permutations and relabel the vertices to their original label.
7. The word produced represents the bipartite graph G, permutationally.

Relabeling Algorithm

1. Given a bipartite graph G, suppose $V_1 = \{a_1, \ldots, a_m\}$ and $V_2 = \{b_1, \ldots, b_n\}$.

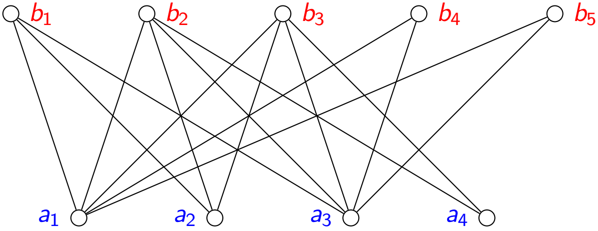
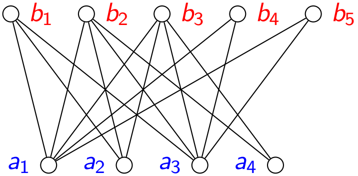


Figure 10 : A bipartite graph G

2. If $N(a) = V_2$ for all $a \in V_1$, then consider the following word and exit:

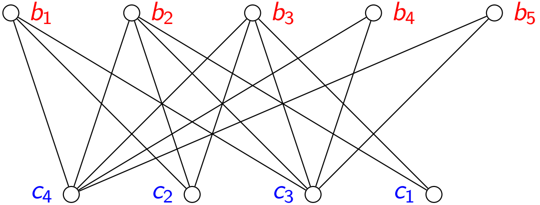
$$w = \overbrace{a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n}\overbrace{a_m a_{m-1} \cdots a_1 b_n b_{n-1} \cdots b_1}.$$

3. Else, choose a in V_1 such that $V_2 \setminus N(a) \neq \varnothing$ and label it as c_1.

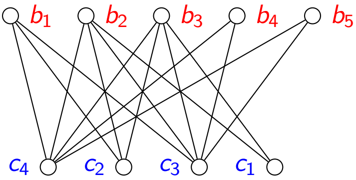


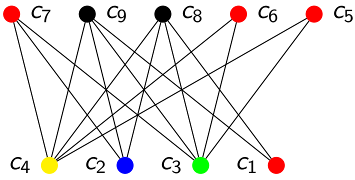
Choosing a to be a_4.

4. Relabel the remaining vertices of V_1 as $c_2, \ldots, c_m$, arbitrarily.



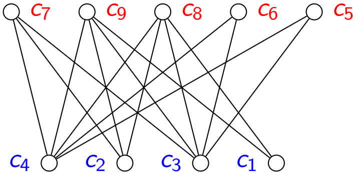
5. Relabel the vertices in V_2 such that the vertices which are not adjacent to c_1 are relabeled first, then the vertices not adjacent to c_2 and so on. Lastly, we relabel the remaining vertices in V_2.





6. Create a list of permutations of the vertices $c_1, \ldots, c_{m+n}$ as per the following:

- $w_1 = c_m c_{m-1} \cdots c_2 c_{m+1} c_{m+2} \cdots c_{m+k_1} c_1 c_{m+k_1+1} c_{m+k_1+2} \cdots c_{m+n}$



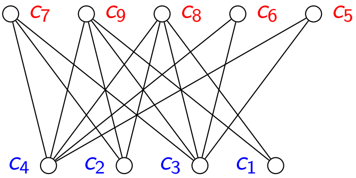
$w_1 = c_4 c_3 c_2 c_5 c_6 c_7 c_1 c_8 c_9$

- For $i = 2$ to m,

  If $V_2 \setminus N(c_i) \neq \varnothing$, then set

  $$w_i = c_1 \cdots c_{i-1} c_{i+1} \cdots c_m \mathrm{Dec}(V_2 \setminus N(c_i)) c_i \mathrm{Dec}(N(c_i));$$

  else, set $w_i = \varepsilon$.

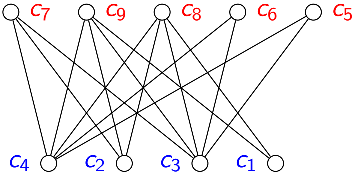


$w_2 = c_1 c_3 c_4 c_6 c_5 c_2 c_9 c_8 c_7$

$w_3 = \varepsilon$

$w_4 = \varepsilon$

7. If $N(c) = V_2$ for some $c \in V_1$ then set
$$w_{m+1} = c_1 c_2 \cdots c_m c_{m+n} \cdots c_{m+1};$$
else, set $w_{m+1} = \varepsilon$.



$$w_5 = c_1 c_2 c_3 c_4 c_9 c_8 c_7 c_6 c_5$$

8. Concatenate the permutations $w_1, w_2, \ldots w_m, w_{m+1}$ to form $w' = w_1 w_2 \cdots w_m w_{m+1}$ which permutationally represents the relabeled graph of G .
$$w' = \overbrace{c_4 c_3 c_2 c_5 c_6 c_7 c_1 c_8 c_9}\,\underbrace{c_1 c_3 c_4 c_6 c_5 c_2 c_9 c_8 c_7}\,\overbrace{c_1 c_2 c_3 c_4 c_9 c_8 c_7 c_6 c_5}$$
9. Replacing the original labels of the vertices of G in the word w'.
$$\overbrace{a_1 a_3 a_2 b_5 b_4 b_1 a_4 b_3 b_2}\,\underbrace{a_4 a_3 a_1 b_4 b_5 a_2 b_2 b_3 b_1}\,\overbrace{a_4 a_2 a_3 a_1 b_2 b_3 b_1 b_4 b_5}$$

Remark

The relabeling algorithm works in $O(mn)$.

Theorem (Correctness of the relabeling algorithm)

The word w generated by the relabeling algorithm represents the bipartite graph G.

Proof: Suppose $G = (V_1 \cup V_2, E)$ such that $m = |V_1| \leq |V_2| = n$. Let G' represent the relabeled graph.
We prove that a and b are adjacent in G' if and only if a and b alternate in the word w'.

- **Case 1:** a and b are adjacent vertices in G'.
  - $w_1 = c_m c_{m-1} \cdots c_2 c_{m+1} c_{m+2} \cdots c_{m+k_1} c_1 c_{m+k_1+1} c_{m+k_1+2} \cdots c_{m+n}$
  - If $w_i \neq \varepsilon$ for $i = 2$ to m,
    $w_i = c_1 \cdots c_{i-1} c_{i+1} \cdots c_m \text{Dec}(V_2 \setminus N(c_i)) c_i \text{Dec}(N(c_i))$
  - If $w_{m+1} \neq \varepsilon$
    $w_{m+1} = c_1 c_2 \cdots c_m c_{m+n} \cdots c_{m+1}$
    else, set $w_{m+1} = \varepsilon$.

In each w_i, the subword ab is produced.

- **Case 2:** a and b are non-adjacent vertices in G'.
  We deal this case in three subcases. In each case, we identify two linear orders (permutations): one with the subword ab and other with the ba.
  - **Subcase 2.1:** $a, b \in V_1$.
    - In w_1, if $a = c_i$ and $b = c_j$ with $i < j$ then ba is a subword.

      $$w_1 = c_m c_{m-1} \cdots c_2 c_{m+1} c_{m+2} \cdots c_{m+k_1} c_1 c_{m+k_1+1} c_{m+k_1+2} \cdots c_{m+n}$$

    - In w_j, if $w_j \neq \varepsilon$ then ab is a subword.

      $$w_j = c_1 \cdots c_{j-1} c_{j+1} \cdots c_m \mathrm{Dec}(V_2 \setminus N(c_j)) c_j \mathrm{Dec}(N(c_j))$$

    - Else, $w_{m+1} \neq \varepsilon$ then ab is a subword.

      $$w_{m+1} = c_1 c_2 \cdots c_m c_{m+n} \cdots c_{m+1}$$

- **Subcase 2.2:** $a \in V_1$ and $b \in V_2$.
  - If $a = c_1$ and $b = c_j$, then ba is a subword of
    $$w_1 = c_m c_{m-1} \cdots c_2 c_{m+1} c_{m+2} \cdots c_{m+k_1} c_1 c_{m+k_1+1} c_{m+k_1+2} \cdots c_{m+n}$$
    The graph has no isolated vertices, $\exists c_k \in V_1$ such that $b \in N(c_k)$,
    $$w_k = c_1 \cdots c_{k-1} c_{k+1} \cdots c_m \text{Dec}(V_2 \setminus N(c_k)) c_k \text{Dec}(N(c_k))$$
    ab is a subword.
  - If $a = c_i$ and $b = c_j$, then ab is a subword of w_1. Since a and b are not adjacent,
    $$w_i = c_1 \cdots c_{i-1} c_{i+1} \cdots c_m \text{Dec}(V_2 \setminus N(c_i)) c_i \text{Dec}(N(c_i))$$
    ba is a subword.

- **Subcase 2.3:** $a, b \in V_2$. If $a = c_i$ and $b = c_j$ with $i < j$. Then ab is a subword of w_1.

$$w_1 = c_m c_{m-1} \cdots c_2 c_{m+1} c_{m+2} \cdots c_{m+k_1} c_1 c_{m+k_1+1} c_{m+k_1+2} \cdots c_{m+n}$$

The graph has no isolated vertices, $\exists c_k \in V_1$ such that $a \in N(c_k)$.

  - If $N(c_k) = V_2$ then $w_{m+1} \neq \varepsilon$, ba is a subword.

$$w_{m+1} = c_1 c_2 \cdots c_m c_{m+n} \cdots c_{m+1}$$

  - Else, $w_k \neq \varepsilon$ then ba is a subword.

$$w_k = c_1 \cdots c_{k-1} c_{k+1} \cdots c_m \mathrm{Dec}(V_2 \setminus N(c_k)) c_k \mathrm{Dec}(N(c_k))$$

Conclusion

Theorem

Let m be the size of the smallest set in the bipartition of a bipartite graph G, the permutation representation number $\mathcal{R}^p(G) \leq m$. Consequently, $\mathcal{R}(G) \leq m$.

Corollary

Let $\{a_{i_1}, \ldots, a_{i_k}\} \subseteq V_1$ be the set of vertices each of which is adjacent to all vertices of V_2 in a bipartite graph G, then $\mathcal{R}^p(G) \leq m - k + 1$.

Remark

The relabeling algorithm proposed for bipartite graphs has a scope to extend it for comparability graphs, in general.

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Thank you.