Subreducts and Subvarieties of PBZ*–lattices

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Bounded lattices with involution: type $(2, 2, 1, 0, 0)$

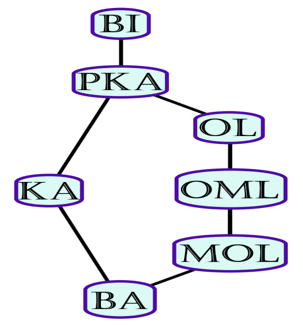
Notation (for any variety $\mathbb{V}$)

$$\Lambda(\mathbb{V}) := \text{the lattice of subvarieties of } \mathbb{V},$$

and its smallest element: $\mathbb{T} := \{A \in \mathbb{V} : |A| = 1\}$

Notation	for the variety of:
$\mathbb{BI}$:	**bounded involution lattices**
$\mathbb{BA}$:	**Boolean algebras**
$\mathbb{PKA}$:	**pseudo–Kleene algebras**
$\mathbb{KA}$:	**Kleene algebras (Kleene lattices)**
$\mathbb{OL}$:	**ortholattices**
$\mathbb{OML}$:	**orthomodular lattices**
$\mathbb{MOL}$:	**modular ortholattices**

Meet–subsemilattice of $\Lambda(\mathbb{BI})$:



Let $(L, \wedge, \vee, \cdot', 0, 1)$, where: $\begin{cases} (L, \wedge, \vee, \cdot', 0, 1): \text{ bounded lattice, with order } \leq \\ \cdot' : L \to L \end{cases}$

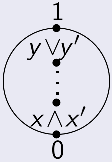
(bounded involution lattice)

$$L \in \mathbb{BI} \iff \begin{cases} L \vDash x'' \approx x \text{ and} \\ L \vDash x \leq y \to y' \leq x' \end{cases} \iff \begin{cases} L \vDash x'' \approx x \text{ and} \\ L \vDash (x \vee y)' \approx x' \wedge y' \text{ and} \\ L \vDash (x \wedge y)' \approx x' \vee y' \end{cases}$$

- $\cdot'$: *involution*

(pseudo–Kleene algebra)

$$L \in \mathbb{PKA} \iff \begin{cases} L \in \mathbb{BI} \text{ and} \\ L \vDash \textbf{the Kleene condition: } x \wedge x' \leq y \vee y' \end{cases}$$



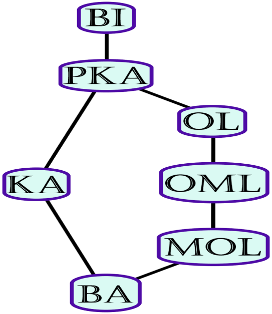
- $\cdot'$: *Kleene complement*

(Kleene algebra)

$$L \in \mathbb{KA} \iff \begin{cases} L \in \mathbb{PKA} \text{ and} \\ L \vDash \textbf{distributivity} \end{cases}$$

(the *(Kleene)–sharp elements* of an $L \in \mathbb{BI}$: $S_K(L)/S(L)$)

$S(L) = \{x \in L \mid x \wedge x' = 0\} = \{x \in L \mid x \vee x' = 1\}$



(ortholattice)

$$L \in \mathbb{OL} \iff \begin{cases} L \in \mathbb{BI} \text{ and} \\ S(L) = L \end{cases}$$

(conditions over $\mathbb{BI}$)

- *paraorthomodularity*: $(x \leq y \ \& \ x' \wedge y \approx 0) \rightarrow x \approx y$

  $\Uparrow \not\Downarrow$ $\Updownarrow$ in $\mathbb{OL}$
- *orthomodularity*: $x \leq y \rightarrow y \approx (x' \wedge y) \vee x$ $(\Leftrightarrow x \vee (x' \wedge (x \vee y)) \approx x \vee y)$

(orthomodular lattice)

$$L \in \mathbb{OML} \iff \begin{cases} L \in \mathbb{BI} \text{ and} \\ L \vDash \textbf{orthomodularity} \end{cases}$$

Bounded lattices with two complements: type $(2, 2, 1, 1, 0, 0)$

(introduced by Roberto GIUNTINI, Antonio LEDDA and Francesco PAOLI for the study of Quantum Logics)

$\mathbb{PBZL}^*$:= the **variety** of **PBZ*–lattices**: the paraorthomodular Brouwer–Zadeh lattices which satisfy condition $(*)$

(PBZ*–lattice)

$$(L, \wedge, \vee, \cdot', \cdot^\sim, 0, 1) \in \mathbb{PBZL}^* \iff \begin{cases} (L, \wedge, \vee, \cdot', 0, 1) \in \mathbb{PKA}, \\ L \vDash \textbf{paraorthomodularity}, \\ \cdot^\sim : L \to L \text{ and } L \text{ satisfies:} \end{cases}$$

$$x \wedge x^\sim \approx 0 \qquad x \leq x^{\sim\sim}$$
$$x \leq y \to y^\sim \leq x^\sim \qquad x^{\sim\prime} \approx x^{\sim\sim}$$
$$(*): \quad (x \wedge x')^\sim \approx x^\sim \vee x'^\sim$$

- $\cdot^\sim$: *Brouwer complement*

(strengthening of condition $(*)$)

- *Strong De Morgan* (**SDM**): $(x \wedge y)^\sim \approx x^\sim \vee y^\sim$

Orthomodular lattices as PBZ*–lattices, and antiortholattices

(for any $L \in \mathbb{PBZL}^*$)

$$S(L) = L \Leftrightarrow (L, \wedge, \vee, \cdot', 0, 1) \in \mathbb{OL} \Leftrightarrow (L, \wedge, \vee, \cdot', 0, 1) \in \mathbb{OML} \Leftrightarrow L \vDash x^{\sim} \approx x'$$

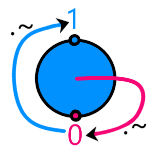
(thus we may identify)

$$\mathbb{OML} \equiv \{L \in \mathbb{PBZL}^* \mid L \vDash x^{\sim} \approx x'\}$$

(antiortholattices: form the positive proper universal class $\mathbb{AOL}$)

$$L \in \mathbb{AOL} \iff \begin{cases} L \in \mathbb{PBZL}^* \text{ and} \\ S(L) = \{0, 1\} \end{cases} \iff \begin{cases} L \in \mathbb{PBZL}^* \text{ and} \\ L \vDash x > 0 \rightarrow x^{\sim} \approx 0 \end{cases}$$

the *trivial* $\cdot^{\sim}$:



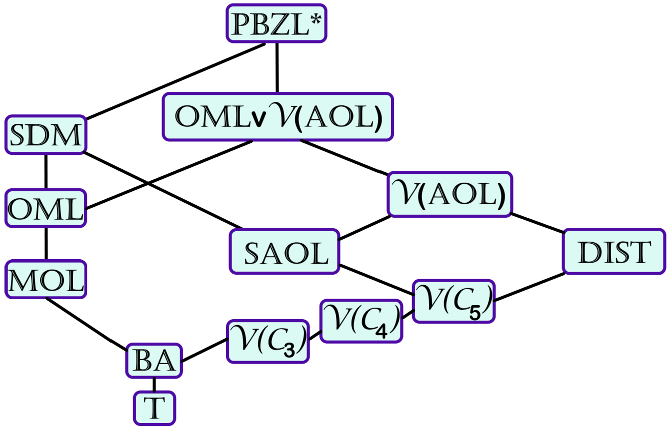
Notation

- $\mathcal{V}(\cdot) \overset{\text{abbreviation}}{=} \mathcal{HSP}(\cdot)$, or $\mathcal{V}_{\mathbb{V}}(\cdot)$ to specify in which variety $\mathbb{V}$
- for all $n \in \mathbb{N}^*$, $\mathcal{C}_n :=$ the n–element chain

(the subvarieties $\mathbb{SDM}$, $\mathbb{SAOL}$ and $\mathbb{DIST}$ of $\mathbb{PBZL}^*$)

- $L \in \mathbb{SDM} \iff \begin{cases} L \in \mathbb{PBZL}^* \text{ and} \\ L \vDash \text{SDM} \end{cases}$
- $\mathbb{SAOL} := \mathbb{SDM} \cap \mathcal{V}(\mathbb{AOL}) = \mathcal{V}(\mathbb{AOL} \cap \mathbb{SDM})$
- $L \in \mathbb{DIST} \iff \begin{cases} L \in \mathbb{PBZL}^* \text{ and} \\ L \vDash \textbf{distributivity} \end{cases}$

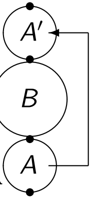
Meet–subsemilattice of $\Lambda(\mathbb{PBZL}^*)$:



A: bounded lattice
$B \in \mathbb{BI}/\mathbb{PKA}$

$A \oplus B \oplus A' \in \mathbb{BI}/\mathbb{PKA}$, respectively
$\Uparrow$

$\cdot'^{A\oplus B\oplus A'}|_A = f$
$\cdot'^{A\oplus B\oplus A'}|_B = \cdot'^{B}$
$\cdot'^{A\oplus B\oplus A'}|_{A'} = f^{-1}$

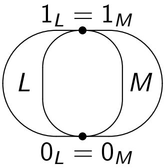


f: dual lattice isomorphism

$|A| > 1$ and $B \in \mathbb{PKA}$
$\Downarrow$
$A \oplus B \oplus A' \in \mathbb{AOL}$

$L, M \in \mathbb{PBZL}^*$, $\quad |L|, |M| > 1$

$\cdot'^{L\boxplus M}|_L = \cdot'^{L}$, $\quad \cdot'^{L\boxplus M}|_M = \cdot'^{M}$
$\cdot^{\sim L\boxplus M}|_L = \cdot^{\sim L}$, $\quad \cdot^{\sim L\boxplus M}|_M = \cdot^{\sim M}$
$\Downarrow$



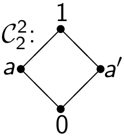
$L \boxplus M \in \mathbb{PBZL}^*$
iff
$L \in \mathbb{OML}$
or
$M \in \mathbb{OML}$

(arbitrary $\boxplus \in \mathbb{PBZL}^*$ iff all but one $\in \mathbb{OML}$)

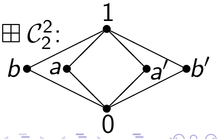
In particular:
$A \oplus A' = A \oplus \mathcal{C}_1 \oplus A'$,
and:

$\mathcal{MO}_0 := \mathcal{C}_2$
for any $\kappa > 0$, $\mathcal{MO}_\kappa := \boxplus_{\iota<\kappa}\mathcal{C}_2^2 \in \mathbb{MOL}$, thus:

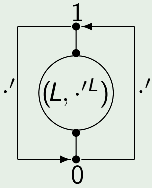
$\mathcal{MO}_1 = \mathcal{C}_2^2$:



$\mathcal{MO}_2 = \mathcal{C}_2^2 \boxplus \mathcal{C}_2^2$:



$(L \in \mathbb{PKA} \implies \mathcal{C}_2 \oplus L \oplus \mathcal{C}_2 \in \mathbb{AOL}$ with the trivial $\cdot^{\sim})$



$$\mathbb{K} \subseteq \mathbb{BI}/\mathbb{PKA}$$
$$\Downarrow$$
$$\mathcal{C}_2 \oplus \mathbb{K} \oplus \mathcal{C}_2 := \{\mathcal{C}_2 \oplus L \oplus \mathcal{C}_2 \mid L \in \mathbb{K}\} \subset \mathbb{BI}/\mathbb{AOL},$$
respectively

(for any $\mathbb{C} \subseteq \mathbb{BI}$ and any $\mathbb{D} \subseteq \mathbb{PKA}$)

$$\mathcal{V}_{\mathbb{BI}}(\mathcal{C}_2 \oplus \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \oplus \mathcal{C}_2) = \mathcal{V}_{\mathbb{BI}}(\mathcal{C}_2 \oplus \mathbb{C} \oplus \mathcal{C}_2) \begin{cases} = \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \Leftrightarrow \mathbb{KA} \subseteq \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \Leftrightarrow \mathcal{C}_3 \in \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \\ \supset \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \Leftrightarrow \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \subseteq \mathbb{OL} \Leftrightarrow \mathcal{C}_3 \notin \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \end{cases}$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathcal{V}_{\mathbb{BI}}(\mathbb{D}) \oplus \mathcal{C}_2) = \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{D} \oplus \mathcal{C}_2)$$

Theorem

The maps:
$$\begin{cases} \Lambda(\mathbb{PKA}) & \longrightarrow & [\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_3), \mathbb{SAOL}]_{\Lambda(\mathbb{PBZL}^*)}, \\ \mathbb{V} & \mapsto & \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) \end{cases}$$
and
$$\begin{cases} \Lambda(\mathbb{PKA}) & \longleftarrow & [\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_3), \mathbb{SAOL}]_{\Lambda(\mathbb{PBZL}^*)}, \\ \{L \in \mathbb{PKA} \mid \mathcal{C}_2 \oplus L \oplus \mathcal{C}_2 \in \mathbb{W}\} & \leftarrow\!\shortmid & \mathbb{W} \end{cases}$$
are mutually inverse lattice isomorphisms.

Bounded involution lattice reducts

Notation

For any $L \in \mathbb{PBZL}^*$ and any $\mathbb{C} \subseteq \mathbb{PBZL}^*$:

- $L_{bi} := (L, \wedge, \vee, \cdot', 0, 1) \in \mathbb{PKA}$
- $\mathbb{C}_{BI} := \{L_{bi} : L \in \mathbb{C}\}$.

($\cdot^\sim$ is unique in any $L \in \mathbb{PBZL}^*$)

For any $L, M \in \mathbb{PBZL}^*$:

- $L = M \iff L_{bi} = M_{bi}$

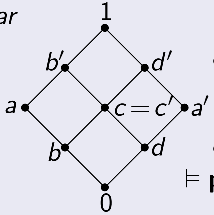
Hence, for any $\mathbb{C}, \mathbb{D} \subseteq \mathbb{PBZL}^*$:

- $\mathbb{C} = \mathbb{D} \iff \mathbb{C}_{BI} = \mathbb{D}_{BI}$
- $\mathbb{C} \subset \mathbb{D} \iff \mathbb{C}_{BI} \subset \mathbb{D}_{BI}$

Corollary

For any $\mathbb{V} \in [\mathbb{SAOL})_{\Lambda(\mathbb{PBZL}^*)}$ (in particular for each $\mathbb{V} \in \{\mathbb{SAOL}, \mathbb{SDM}, \mathbb{PBZL}^*\}$):

- $\mathcal{V}_{\mathbb{BI}}(\mathbb{V}_{BI}) = \mathbb{PKA}$;
- $\mathbb{V}_{BI}$ *is not a variety;*
- $\mathbb{V}_{BI} \subseteq \mathbb{PBZL}^*_{BI} \subset \{L \in \mathbb{PKA} : L \vDash$ **paraorthomodularity**$\}$.



$\notin \mathbb{PBZL}^*_{BI}$

$\in \mathbb{PKA}$

$\vDash$ **paraorthomodularity**

Axiomatizations for $\mathbb{V} \in \Lambda(\mathbb{PKA}) \rightsquigarrow \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2)$

- $k, n, p \in \mathbb{N}$
- $t(x_1, \ldots, x_k, z_1, \ldots, z_p)$, $u(y_1, \ldots, y_n, z_1, \ldots, z_p)$: terms over $\mathbb{BI}$

(terms over $\mathbb{PBZL}^*$)

$$m(t,u)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) =$$
$$\bigvee_{i=1}^{k}(x_i \wedge x_i')^{\sim} \vee \bigvee_{j=1}^{n}(y_j \wedge y_j')^{\sim} \vee \bigvee_{h=1}^{p}(z_h \wedge z_h')^{\sim} \vee t(x_1, \ldots, x_k, z_1, \ldots, z_p).$$

$$\Downarrow$$

$$m(u,t)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) =$$
$$\bigvee_{i=1}^{k}(x_i \wedge x_i')^{\sim} \vee \bigvee_{j=1}^{n}(y_j \wedge y_j')^{\sim} \vee \bigvee_{h=1}^{p}(z_h \wedge z_h')^{\sim} \vee u(y_1, \ldots, y_n, z_1, \ldots, z_p).$$

Lemma

- $L \in \mathbb{BI}$, $\mathcal{C}_3 \in V_{\mathbb{BI}}(L) \Longrightarrow [L \vDash t \approx u \Leftrightarrow \mathcal{C}_2 \oplus L \oplus \mathcal{C}_2 \vDash t \approx u]$
- $L \in \mathbb{PKA}$, $k+p>0<n+p \Longrightarrow [L \vDash t \approx u \Leftrightarrow \mathcal{C}_2 \oplus L \oplus \mathcal{C}_2 \vDash m(t,u) \approx m(u,t)]$

For any $L \in \mathbb{BI} \supset \mathbb{PKA}$:

(the ortholattice equation $x \wedge x' \approx 0$ written with nonnullary terms)

$L \in \mathbb{OL} \iff L \vDash x \wedge x' \approx y \wedge y' \iff L \vDash x \vee x' \approx y \vee y'$

(the equations $m(t,u) \approx m(u,t)$ for $t \approx u$ the $\mathbb{OL}$ equations above)

- D2OL$\wedge$: $(x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee (x \wedge x') \approx (x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee (y \wedge y')$
- D2OL$\vee$: $(x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee x \vee x' \approx (x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee y \vee y'$

$\mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL}) = \{L \in \mathbb{PBZL}^* \mid L \vDash x \approx (x \wedge y^{\sim}) \vee (x \wedge y^{\sim\sim})\}$

Theorem

$\mathbb{V}$ subvariety of $\mathbb{PKA}$; $\{t_i, u_i \mid i \in I\}$ terms over $\mathbb{BI}$.

- *If $\mathcal{C}_3 \in \mathbb{V}$ (equivalently, if $\mathbb{KA} \subseteq \mathbb{V}$), then:*

$$\mathbb{V} = \{L \in \mathbb{PKA} \mid (\forall\, i \in I)\,(L \vDash t_i \approx u_i)\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathbb{SAOL} \mid (\forall\, i \in I)\,(L \vDash t_i \approx u_i)\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL}) \mid L \vDash \text{SDM},\ (\forall\, i \in I)\,(L \vDash t_i \approx u_i)\}$$

$$\iff \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathbb{PBZL}^* \mid L \vDash x \approx (x \wedge y^{\sim}) \vee (x \wedge y^{\sim\sim}),$$

$$L \vDash \text{SDM},\ (\forall\, i \in I)\,(L \vDash t_i \approx u_i)\}.$$

- If $\mathcal{C}_3 \notin \mathbb{V}$ (equivalently, if $\mathbb{V} \subseteq \mathbb{OL}$) and, for all $i \in I$, t_i and u_i have nonzero arities, then:

$$\mathbb{V} = \{L \in \mathbb{OL} \mid (\forall\, i \in I)\,(L \vDash t_i \approx u_i)\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL}) \mid L \vDash \mathrm{D2OL}\wedge,$$

$$(\forall\, i \in I)\,(L \vDash m(t_i, u_i) \approx m(u_i, t_i))\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL}) \mid L \vDash \mathrm{D2OL}\vee,$$

$$(\forall\, i \in I)\,(L \vDash m(t_i, u_i) \approx m(u_i, t_i))\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathbb{PBZL}^* \mid L \vDash x \approx (x \wedge y^{\sim}) \vee (x \wedge y^{\sim\sim}),$$

$$L \vDash \mathrm{D2OL}\wedge,\ (\forall\, i \in I)\,(L \vDash m(t_i, u_i) \approx m(u_i, t_i))\} \iff$$

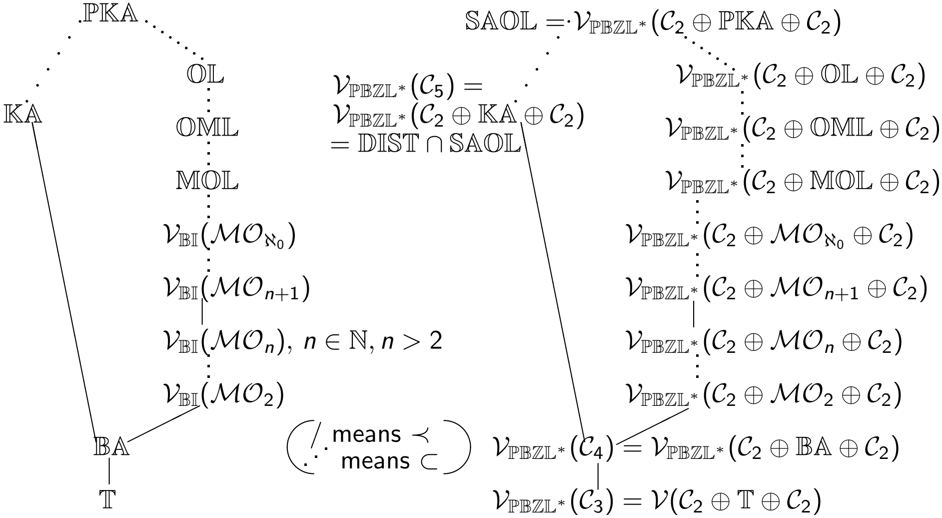
$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathbb{PBZL}^* \mid L \vDash x \approx (x \wedge y^{\sim}) \vee (x \wedge y^{\sim\sim}),$$

$$L \vDash \mathrm{D2OL}\vee,\ (\forall\, i \in I)\,(L \vDash m(t_i, u_i) \approx m(u_i, t_i))\}.$$

So we have two different ways of determining an axiomatization for $\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2)$ from one for $\mathbb{V}$, depending on whether $\mathbb{V}$ belongs to the filter $[\mathbb{KA})$ or the ideal $(\mathbb{OL}]$ determined by the splitting pair $(\mathbb{OL}, \mathbb{KA})$ in the lattice of subvarieties of $\mathbb{PKA}$.

Mapping the infinite ascending chain from $\Lambda(\mathbb{MOL})$

The lattice isomorphism $\Lambda(\mathbb{PKA}) \cong [\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_3), \mathbb{SAOL}]_{\Lambda(\mathbb{PBZL}^*)}$ above restricts to a poset isomorphism between:



Recall, also, that $\mathbb{OML} \subset \mathbb{PBZL}^*$.

We have obtained the infinite ascending chains of subvarieties of $\mathbb{PBZL}^*$:

- $\mathcal{B} := \{\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{MO}_n) : n \in (\mathbb{N} \setminus \{0,1\}) \cup \{\aleph_0\}\} \subset \Lambda(\mathbb{MOL}) \subset \Lambda(\mathbb{OML}) \subset \Lambda(\mathbb{PBZL}^*)$
- $\mathcal{C} := \{\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathcal{MO}_n \oplus \mathcal{C}_2) : n \in (\mathbb{N} \setminus \{0,1\}) \cup \{\aleph_0\}\} \subset \Lambda(\mathbb{SAOL}) \subset \Lambda(\mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL})) \subset \Lambda(\mathbb{PBZL}^*)$

Theorem

The following is an infinite ascending chain of subvarieties of $\mathbb{DIST}$*:*

$$\mathbb{T} = V_{\mathbb{PBZL}^*}(\mathcal{C}_1) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^0 \oplus \mathcal{C}_2^0) \prec \mathbb{BA} = V_{\mathbb{PBZL}^*}(\mathcal{C}_2) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^0 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^0) \prec$$

$$V_{\mathbb{PBZL}^*}(\mathcal{C}_3) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^1 \oplus \mathcal{C}_2^1) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_4) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^1) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_5)$$

$$\prec V_{\mathbb{PBZL}^*}(\mathcal{C}_2^2 \oplus \mathcal{C}_2^2) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_2^2 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^2) \prec \ldots \prec$$

$$V_{\mathbb{PBZL}^*}(\mathcal{C}_2^n \oplus \mathcal{C}_2^n) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_2^n \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^n) \prec$$

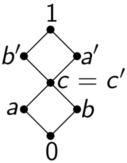
$$V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{n+1} \oplus \mathcal{C}_2^{n+1}) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{n+1} \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^{n+1}) \prec \ldots \subset$$

$$V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{\aleph_0} \oplus \mathcal{C}_2^{\aleph_0}) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{\aleph_0} \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^{\aleph_0}) \subseteq$$

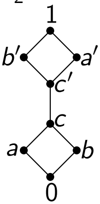
$$V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{\kappa} \oplus \mathcal{C}_2^{\kappa}) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{\kappa} \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^{\kappa}) \subseteq \mathbb{DIST},$$

where $n \in \mathbb{N} \setminus \{0,1,2\}$ *and* κ *is an infinite cardinality.*

$\mathcal{C}_2^2 \oplus \mathcal{C}_2^2 \in \mathbb{DIST} \cap \mathbb{AOL}$:



$\mathcal{C}_2^2 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^2 \in \mathbb{DIST} \cap \mathbb{AOL}$:



So we also have the infinite ascending chain of subvarieties of $\mathbb{PBZL}^*$:

- $\mathcal{D} := \{\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2^n \oplus \mathcal{C}_2^n), \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2^n \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^n) : n \in (\mathbb{N} \setminus \{0,1\}) \cup \{\aleph_0\}\} \subset \Lambda(\mathbb{SAOL}) \subset \Lambda(\mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL})) \subset \Lambda(\mathbb{PBZL}^*)$

Consequently, for any $\mathbb{U}, \mathbb{X} \in \Lambda(\mathbb{OML}) \setminus \{\mathbb{T}\}$ such that $\mathbb{U} \neq \mathbb{X}$ and any $\mathbb{V}, \mathbb{W} \in \Lambda(\mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL})) \setminus \{\mathbb{T}\}$ such that $\mathbb{V} \neq \mathbb{W}$:

$\{\mathbb{Y} \vee \mathbb{V} : \mathbb{Y} \in \mathcal{B}\}$, $\{\mathbb{Y} \vee \mathbb{W} : \mathbb{Y} \in \mathcal{B}\}$, $\{\mathbb{U} \vee \mathbb{Y} : \mathbb{Y} \in \mathcal{C}\}$, $\{\mathbb{X} \vee \mathbb{Y} : \mathbb{Y} \in \mathcal{C}\}$, $\{\mathbb{U} \vee \mathbb{Y} : \mathbb{Y} \in \mathcal{D}\}$ and $\{\mathbb{X} \vee \mathbb{Y} : \mathbb{Y} \in \mathcal{D}\}$ are pairwise disjoint infinite ascending chains in $\Lambda(\mathbb{OML} \vee \mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL})) \subset \Lambda(\mathbb{PBZL}^*)$.

C. Mureşan, Subreducts and Subvarieties of PBZ*–lattices, arXiv:1904.10093v4 [math.RA].

THANK YOU FOR YOUR ATTENTION !