Many-valued modal logic over a semi-primal algebra

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Joint work (in progress) with Alexander Kurz and Bruno Teheux

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Semi-primal algebras

The two-element Boolean algebra **2** is *primal*. That is, every map $f : 2^n \to 2$ is term-definable.

Definition (Foster, Pixley 1964)

A finite algebra $\boldsymbol{L}$ is *semi-primal* if every map $f : L^n \to L$ which preserves subalgebras is term-definable.

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In a *discriminator algebra* the ternary discriminator is term-definable:

$$t(x, y, z) = \begin{cases} z & \text{if } x = y \\ x & \text{if } x \neq y \end{cases}$$

Proposition (Pixley 1971)

A finite algebra $\boldsymbol{L}$ is semi-primal if and only if it is a discriminator algebra and it has no internal isomorphisms.

Semi-primal algebras with bounded lattice reduct

In this talk we exclusively consider semi-primal algebras $\boldsymbol{L}$ with an underlying bounded lattice structure $\boldsymbol{L}^\flat = (L, \wedge, \vee, 0, 1)$.

Proposition

Let $\boldsymbol{L}$ be a finite algebra with bounded lattice reduct. Then $\boldsymbol{L}$ is semi-primal if and only if for every $a \in L$ the map $T_a : L \to L$:

$$T_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

is term-definable.

Examples

- Chain-based: The $(n+1)$-element Łukasiewicz-chain

$$\text{Ł}_n = (\{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, \oplus, \neg, 0, 1)$$

where

$$x \oplus y = min(x+y, 1) \text{ and } \neg x = 1 - x.$$

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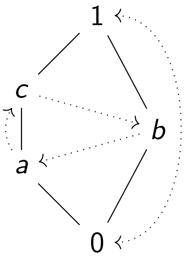
- Not chain-based: $\boldsymbol{L} = (N_5, \wedge, \vee, ', 0, 1)$:



(Davey, Schumann, Werner 1991)

Stone duality and semi-primal natural duality

For $\mathsf{BA} = \mathbb{ISP}(\mathbf{2})$ we have the famous *Stone duality*:

$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

$\Sigma(\boldsymbol{B}) = \mathsf{BA}(\boldsymbol{B}, \mathbf{2})$ and $\Pi(X) = \mathsf{Stone}(X, 2)$.

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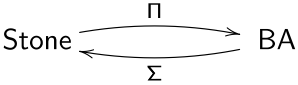
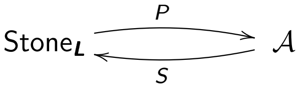


$\Sigma(\boldsymbol{B}) = \mathsf{BA}(\boldsymbol{B}, \mathbf{2})$ and $\Pi(X) = \mathsf{Stone}(X, 2)$.

For $\mathcal{A} := \mathbb{ISP}(\boldsymbol{L})$ we have the similar *natural duality*:



"$S(\boldsymbol{A}) = \mathcal{A}(\boldsymbol{A}, \boldsymbol{L})$ and $P(X) = \mathsf{Stone}_{\boldsymbol{L}}(X, L)$"
(Keimel, Werner 1973)

The category $\mathsf{Stone}_{\boldsymbol{L}}$

The category $\mathsf{Stone}_{\boldsymbol{L}}$ has objects (X, v) where

- $X \in \mathsf{Stone}$,
- $v : X \to \mathbb{S}(\boldsymbol{L})$,
- $v^{-1}(\boldsymbol{S}\downarrow)$ is closed for every subalgebra $\boldsymbol{S} \leq \boldsymbol{L}$.

A morphism $f : (X, v) \to (Y, v')$ is a continuous map with

$$v'(f(x)) \subseteq v(x)$$

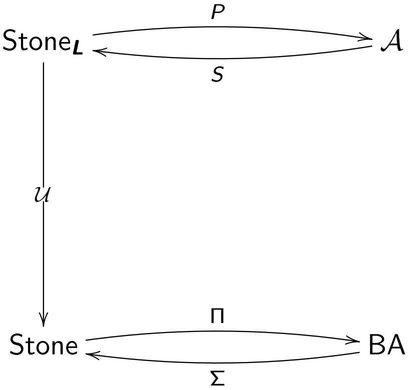
for every $x \in X$.

Relating the dualities

$$\mathsf{Stone}_{\boldsymbol{L}} \underset{S}{\overset{P}{\rightleftarrows}} \mathcal{A}$$

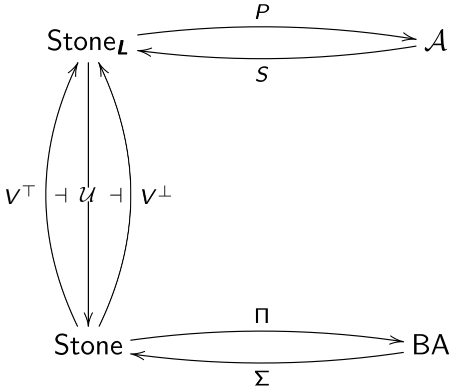
$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

Relating the dualities



Let $\mathcal{U}$ be the forgetful functor.

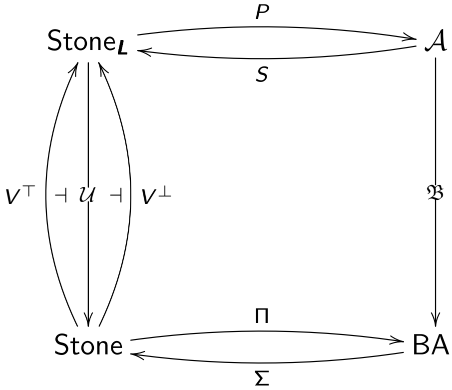
Relating the dualities



Let $\mathcal{U}$ be the forgetful functor.

$V^\top(X) = (X, v^\top)$ where $v^\top(x) = L$ for all $x \in X$.
$V^\perp(X) = (X, v^\perp)$ where $v^\perp(x) = 2$ for all $x \in X$.

Relating the dualities



Let $\mathcal{U}$ be the forgetful functor.

$V^\top(X) = (X, v^\top)$ where $v^\top(x) = L$ for all $x \in X$.
$V^\perp(X) = (X, v^\perp)$ where $v^\perp(x) = \langle 0, 1\rangle$ for all $x \in X$.

The Boolean skeleton

Recall: For every $a \in L$ the map $T_a : L \to L$ given by

$T_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$ is term-definable in $\boldsymbol{L}$.

Definition (Maruyama 2011)

Given any $\boldsymbol{A} \in \mathcal{A}$ we define $\mathfrak{B}(A) = \{a \in A \mid T_1(a) = a\}$. The *Boolean skeleton* of $\boldsymbol{A}$ is the Boolean algebra

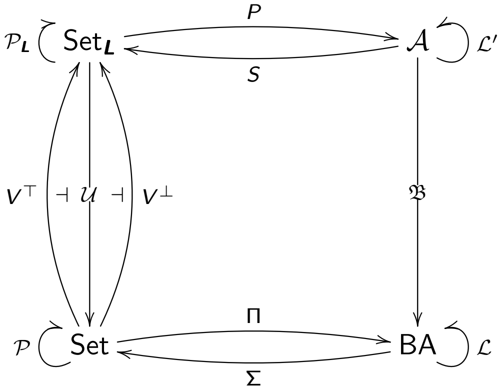
$$\mathfrak{B}(\boldsymbol{A}) = (\mathfrak{B}(A), \wedge, \vee, T_0, 0, 1).$$

Theorem

Let $\boldsymbol{A} \in \mathcal{A}$. Then $h \mapsto h\restriction_{\mathfrak{B}(A)}$ is a homeomorphism

$$\mathcal{A}(\boldsymbol{A}, \boldsymbol{L}) \simeq \mathsf{BA}(\mathfrak{B}(\boldsymbol{A}), \mathbf{2}).$$

Going modal, coalgebraically



$\mathcal{P}$-coalgebras: (Kripke-)frames
$\mathcal{L}$-algebras: Classical modal logic

$\mathcal{P}'$-coalgebras: $\boldsymbol{L}$-(Kripke-)frames
$\mathcal{L}'$-algebras: Modal logic over $\boldsymbol{L}$

Relational semantics

Definition

An $\boldsymbol{L}$-*frame* is a triple (W, R, v) such that

- (W, R) is a Kripke-frame,
- $(W, v) \in \mathrm{Set}_{\boldsymbol{L}}$, (that is, $v : W \to \mathbb{S}(\boldsymbol{L})$) and
- Compatibility: $wRw' \Rightarrow v(w') \subseteq v(w)$.

An $\boldsymbol{L}$-*model* is given by $\mathfrak{M} = (W, R, v, Val)$ such that (W, R, v) is an $\boldsymbol{L}$-frame and

$$Val : W \times \mathrm{Prop} \to L$$

satisfies for all $w \in W$, $p \in \mathrm{Prop}$:

$$Val(w, p) \in v(w).$$

We can extend Val to all modal formulas and define the validity relation $\mathfrak{M}, w \models \varphi$ iff $Val(w, \varphi) = 1$.

The '$\boldsymbol{L}$-versions' of the Goldblatt-Thomason theorem

The functor $\mathfrak{B}$ - Boolean skeleton - is used in the proof of the following:

Theorem

Let F be a class of $\boldsymbol{L}$-frames closed under ultrapowers. Then F is definable if and only if F is closed under disjoint unions, generated subalgebras and bounded morphic images, and it reflects canonical extensions.

The dually adjoint functor $V^{\top}$ is then used in the proof of the following:

Corollary

Let F be a class of Kripke-frames closed under ultrapowers. Then F is $\boldsymbol{L}$-definable if and only if F is closed under disjoint unions, generated subalgebras and bounded morphic images, and it reflects ultrafilter extensions.

Thanks for your attention!