ℓr-Multisemigroups, Modal Quantales and the Origin of Locality

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Motivation

a quantale $(Q, \leq, \cdot, 1)$ is a complete lattice $(Q, \leq)$ and
a monoid $(Q, \cdot, 1)$, and $\cdot$ preserves sups in both arguments

for $f, g : X \to Q$ from relational structure (X, R) with ternary R
a convolution operation is defined as

$$(f * g)(x) = \bigvee_{R(x,y,z)} f(y) \cdot g(z)$$

the convolution algebra is the algebra on Q^X

Convolution as a binary modality

$$(f * g)(x) = \bigvee_{R(x,y,z)} f(y) \cdot g(z) \qquad\qquad f, g : X \to Q$$

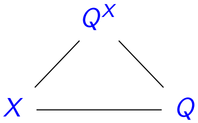
in Lambek calculus, convolution is a binary modality over a ternary frame

in boolean algebras with operators
n-ary modalities in B are dual to $n+1$-ary relations in X

there are correspondences between properties in X and identities in B

Correspondences for Convolution Algebras

for general Q we get correspondence triangles



these yield uniform construction recipes for convolution algebras Q^X from X and value algebras Q

today: find X corresponding to modal quantale $\mathbb{B}^X$, with view on locality

Modal quantales

a domain quantale is a quantale with $dom : Q \to Q$ satisfying

$$dom(\alpha) \cdot \alpha = \alpha \qquad \text{(absorption)}$$
$$dom(\alpha \vee \beta) = dom(\alpha) \vee dom(\beta) \qquad \text{(finite sup-preservation)}$$
$$dom(\bot) = \bot \qquad \text{(empty sup-preservation)}$$
$$dom(\alpha) \leq 1 \qquad \text{(subidentity)}$$
$$dom(\alpha \cdot \beta) = dom(\alpha \cdot dom(\beta)) \qquad \text{(locality)}$$

a codomain quantale (Q, cod) is a domain quantale in (Q^{op}, dom)

a modal quantale is a domain and codomain quantale such that

$$dom \circ cod = cod \qquad cod \circ dom = dom \qquad \text{(compatibility)}$$

if $\mathbb{B}^X$ is a modal quantale ... which structure has X?

Object-free category?

categories. A category is a set C of arrows with two functions $s, t : C \to C$, called "source" and target", and a partially defined binary operation $\#$, called composition, all subject to the following axioms, for all x, y, and z in C:

The operation $x \# y$ is defined iff $s\,x = t\,y$ and then

$$s(x \# y) = s\,y\,, \qquad t\,(x \# y) = t\,x\,; \tag{1}$$

$$x \# s\,x = x\,, \qquad t\,x \# x = x\,; \tag{2}$$

$$(x \# y) \# z = x \# (y \# z) \quad \text{if either side is defined}\,; \tag{3}$$

$$s\,s\,x = s\,x = t\,s\,x\,;$$

$$t\,t\,x = t\,x = s\,t\,x\,. \tag{4}$$

Then x is an identity iff $x = s\,x$ or, equivalently, iff $x = t\,x$.

[MacLane, Ch.XII.5]

we want to be more general/relational

Multioperations

$$\mathcal{P}(X \times X \times X) \;\simeq\; X \times X \to \mathcal{P}X$$

a multioperation is a function $X \times X \to \mathcal{P}X$

we extend $\odot : X \times X \to \mathcal{P}X$ to

$$\odot : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X,\ (A, B) \mapsto \bigcup \{x \odot y \mid x \in A, y \in B\}$$

$\odot$ is a partial operation if $|x \odot y| \leq 1$ for all $x, y \in X$

$\odot$ is a (total) operation if $|x \odot y| = 1$ for all $x, y \in X$

the shuffle of words is a multioperation

ℓr-Multisemigroups

an ℓr-multimagma is a multimagma $(X, \odot)$ with $\ell, r : X \to X$ satisfying

$$x \odot y \neq \emptyset \Rightarrow r(x) = \ell(y)$$

$$\ell(x) \odot x = \{x\} \qquad x \odot r(x) = \{x\} \qquad \text{(absorption)}$$

an ℓr-multisemigroup is an associative ℓr-multimagma

$$x \odot (y \odot z) = (x \odot y) \odot z$$

an ℓr-multimagma is ℓr-local if $r(x) = \ell(y) \Rightarrow x \odot y \neq \emptyset$

categories are precisely the partial ℓr-local ℓr-semigroups

ℓr-locality captures the composition pattern of categories

Examples

pair groupoid $(A \times A, \odot, \ell, r)$ is an ℓr-local partial ℓr-semigroup with

$$(a, b) \odot (c, d) = \begin{cases} \{(a, d)\} & \text{if } b = c \\ \emptyset & \text{otherwise} \end{cases} \qquad \ell((a, b)) = (a, a) = r((b, a))$$

shuffle ℓr-multisemigroup $(A^*, \|, \ell, r)$ is ℓr-local because $\|$ is total with $\ell(w) = \varepsilon = r(w)$

PAMs with unit 1 used in separation logic are non-local because $\ell(x) = 1 = r(x)$ and composition is partial

paths $f : [0, 1] \to T$ in topology form local partial ℓr-magmas

more generally, elements of $X_\ell = \{x \mid \ell(x) = x\} = \{x \mid r(x) = x\} = X_r$ are orthogonal idempotent units of X $(\ell(x) \neq \ell(y) \Leftrightarrow \ell(x)\ell(y) = \emptyset)$

Properties

in ℓr-multimagmas

$$\ell \circ r = r \qquad r \circ \ell = \ell \qquad \text{(compatibility)}$$

$$\ell(\ell(x)y) = \ell(x)\ell(y) \qquad r(xr(y)) = r(x)r(y) \qquad \text{(export)}$$

in ℓr-multisemigroups

$$\ell(xy) \subseteq \ell(x\ell(y)) \qquad r(xy) \subseteq r(r(x)y) \qquad \text{(weak locality)}$$

in ℓr-local ℓr-multisemigroups

$$\ell(xy) = \ell(x\ell(y)) \qquad r(xy) = r(r(x)y) \qquad \text{(locality)}$$

Origin of locality

ℓr-multisemigroup is ℓr-local iff

$$\ell(x\ell(y)) = \ell(xy) \qquad r(r(x)y) = r(xy)$$

equational locality thus captures the composition pattern of categories

$$xy \neq \emptyset \Leftrightarrow r(x) = \ell(y)$$

Constructing powerset quantales

if $(X, \odot, \ell, r)$ is an ℓr-multisemigroup, then $(\mathcal{P}X, \subseteq, \odot, X_\ell)$ is a boolean quantale whose underlying lattice is atomic

all $\ell(x)/r(x)$ in X are combined into unit X_ℓ of $\mathcal{P}X$

categories lift to powerset quantales with arrows as atoms

pair groupoids lift to quantales of binary relations

we refine this construction lifting $dom = \mathcal{P}\ell$ and $cod = \mathcal{P}r$

Lifting to modal powerset quantales

from ℓr-multimagma

$$\ell(A)A = A \qquad Ar(A) = A \qquad \text{(absorption)}$$

$$\ell\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} \ell(A) \qquad r\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} r(A) \qquad \text{(sup-preservation)}$$

$$\ell(A) \subseteq X_\ell \qquad r(A) \subseteq X_r \qquad \text{(subidentity)}$$

$$\ell(r(A)) = r(A) \qquad r(\ell(A)) = \ell(A) \qquad \text{(compatibility)}$$

$$\ell(\ell(A)B) = \ell(A)\ell(B) \qquad r(Ar(B)) = r(A)r(B) \qquad \text{(export)}$$

from ℓr-multisemigroup

$$\ell(AB) \subseteq \ell(A\ell(B)) \qquad r(AB) \subseteq r(r(A)B) \qquad \text{(weak locality)}$$

from local ℓr-multisemigroup

$$\ell(AB) = \ell(A\ell(B)) \qquad r(AB) = r(r(A)B) \qquad \text{(locality)}$$

Modal correspondences

if X is an ℓr-multimagma,

- then $(\mathcal{P}X, \subseteq, \odot, X_\ell, dom, cod)$ is a boolean modal prequantale
- it is a weakly local modal quantale if X is an ℓr-multisemigroup
- it is a modal quantale if X is a local ℓr-multisemigroup

we get converse directions, too

- if $\mathcal{P}X$ is a prequantale, then X is an ℓr-multimagma
- if $\mathcal{P}X$ is a quantale, then X is an ℓr-multisemigroup
- if $\mathcal{P}X$ is a modal quantale, then X is a local ℓr-multisemigroup

we don't see $xy \neq \emptyset \Leftrightarrow r(x) = \ell(y)$ in quantale
instead we see $\alpha\beta \neq 0 \Leftrightarrow cod(\alpha)dom(\beta) \neq 0$

Modal Correspondences

$$
\begin{aligned}
dom(A \odot dom(B)) &= \bigcup\{\ell(x \odot \ell(y)) \mid x \in A,\ y \in B, r(x) = \ell(\ell(y))\} \\
&= \bigcup\{\ell(x \odot y) \mid x \in A,\ y \in B, r(x) = \ell(y)\} \\
&= dom(A \odot B)
\end{aligned}
$$

$$
\begin{aligned}
\ell(x \odot \ell(y)) &= dom(\{x\} \odot dom(\{y\})) \\
&= dom(\{x\} \odot \{y\}) \\
&= \ell(x \odot y)
\end{aligned}
$$

Examples

categories lift to modal powerset quantales

pair groupoids lift to modal powerset quantales of binary relations

shuffle multisemigroups lift to quantales of shuffle languages

PAMs lift to non-local assertion quantales of separation logic

path algebras in topology lift to modal powerset prequantales

Jónsson/Tarski knew that groupoids lift to RAs with converse

Discussion

further examples can be found in paper

results extend to convolution algebras Q^X with correspondence triangles

Q can be semiring/Kleene algebra when finiteness properties for $\bigvee$ hold

see arXiv paper for details

Conclusion

we introduced ℓr-multisemigroups

related them with categories

showed how ℓ/r correspond to dom/cod

explained how locality relates to composition pattern of categories

presented generic construction recipe for modal powerset quantales