Skew metrics (linear orders) valued in Sugihara semigroups

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Plan

Linear orders valued in Girard quantales

Linear orders valued in Sugihara monoids

Generalizing linear orders

A *strict linear order* is a transitive tournament.

The characteristic function

$$\chi_\ell : \{ (x, y) \mid x \neq y \} \longrightarrow \mathbf{2}$$

of a linear order ℓ on X satisfies

$$\chi_\ell(x, y) \wedge \chi_\ell(y, z) \leq \chi_\ell(x, z) \quad \text{and} \quad \chi_\ell(y, x) = \neg\chi_\ell(x, y) \,.$$

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$$X^2_{\mathrm{s}} := \{ (x, y) \mid x \neq y \}\,.$$

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Linear orders on ordered semigroups

Let $\mathcal{Q} = (Q, \leq, \otimes)$ be an ordered semigroup coming with an antitone involution $^\star : Q \longrightarrow Q$.

Definition. A (strict) *linear order* valued in $\mathcal{Q}$ is a function

$$\ell : X_s^2 \longrightarrow \mathcal{Q}$$

such that

$$\ell(x, y) \otimes \ell(y, z) \leq \ell(x, z) \quad \text{and} \quad \ell(y, x) = \ell(x, y)^\star .$$

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Remark. Such a linear order is also a *skew metric*, meaning that

$$\ell(x, z) \leq \ell(x, y) \oplus \ell(y, z) \quad \text{and} \quad \ell(y, x) = \ell(x, y)^\star ,$$

with

$$x \oplus y := (y^\star \otimes x^\star)^\star .$$

Linear orders and clopens

Suppose that X is totally ordered (e.g. if $X = \{1, \dots, n\}$).
Let $\mathcal{I}_X := \{ (x, y) \mid x < y \}$.

A map $f : \mathcal{I}_X \longrightarrow \mathcal{Q}$ is

- closed (transitive) if

$$f(x, y) \otimes f(y, z) \leq f(x, z)$$

- open (cotransitive) if

$$f(x, z) \leq f(x, y) \oplus f(y, z)$$

- clopen if it is closed and open.

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The restriction of a linear order $\ell : X_s^2 \longrightarrow \mathcal{Q}$ to $\mathcal{I}_X$ is clopen.

Proposition. If $\mathcal{Q}$ is a Girard quantale, then every clopen map $f : \mathcal{I}_X \longrightarrow Q$ uniquely extends to a linear order valued in $\mathcal{Q}$.

Girard quantales

Definition. A *Girard quantale* (also, an involutive residuated lattice, ...) is a tuple $\mathcal{Q} = (Q, \leq, \otimes, ^\star)$ with

- $(Q, \leq)$ a complete lattice,
- $(Q, \leq, \otimes)$ is an ordered semigroup,
- $^\star : Q \longrightarrow Q$ is an antitone involution

such that

$$x \otimes y \leq z \quad \text{iff} \quad y \otimes z^\star \leq x^\star \quad \text{iff} \quad z^\star \otimes x \leq y^\star \,. \tag{1}$$

for each $x, y, z \in Q$.

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for each $x, y, z \in Q$.

Equivalently to (1)

$$x \otimes y \leq z^\star \quad \text{iff} \quad x \leq (y \otimes z)^\star \quad \text{iff} \quad y \leq (z \otimes x)^\star \,.$$

Examples I

- **2**, the standard Boolean algebra.
  - Linear orders are linear orders (permutations).

- **3**, the Sugihara monoid on the 3-element chain. That is:

$\otimes$	-1	0	1
-1	-1	-1	-1
0	-1	0	1
1	-1	1	1

$\oplus$	-1	0	1
-1	-1	-1	1
0	-1	0	1
1	1	1	1

  - Linear orders $\ell : [n]_s^2 \longrightarrow \mathbf{3}$ are linear preorders (pseudo-permutations).

Examples II

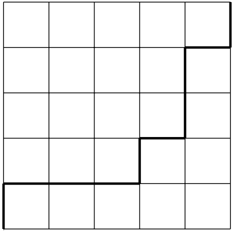
- $\mathcal{Q}_k$ is the collection of sup-preserving functions

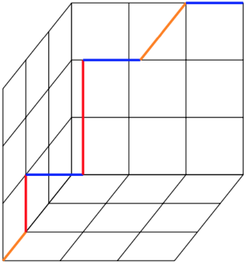
$$f : \{0, 1, \ldots, k\} \longrightarrow \{0, 1, \ldots, k\}$$

ordered pointwise, $\otimes$ is function composition.

Equivalently, elements of $\mathcal{Q}_k$ are
  - discrete paths from $(0, 0)$ to (k, k) made up of East/North steps,
  - the ordering is the dominance ordering,
  - $\star$ is reflection along the diagonal.

• Linear orders $\ell : [n]^2_s \longrightarrow \mathcal{Q}_k$ are n-dimensional paths from $(0, \ldots, 0)$ to $(k, \ldots, k)$ (or words on the alphabet $\{x_1, \ldots, x_n\}$ with k occurrences of each letter x_i).





Examples III

- $\mathcal{Q}_{[0,1]}$ is the collection of sup-preserving functions

$$f : [0,1] \longrightarrow [0,1]$$

ordered pointwise, $\otimes$ is function composition.

Equivalently, elements of $\mathcal{Q}_{[0,1]}$ are

  - (images of) continuous monotonic paths from $(0,0)$ to $(1,1)$,
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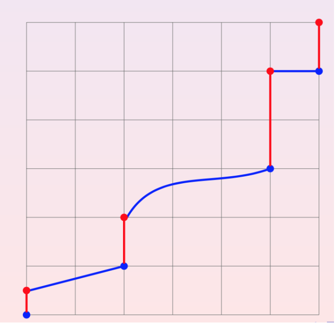
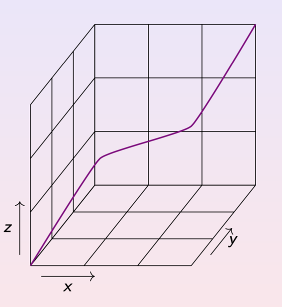
• Linear orders $\ell : [n]^2_s \longrightarrow \mathcal{Q}_{[0,1]}$ are (images of) continuous increasing n-dimensional paths from $(0,\ldots,0)$ to $(1,\ldots,1)$. (RAMICS 2018)




z
y
x

Linear orders form a lattice

We suppose X is linearly ordered. For linear orders $f, g : X_s^2 \longrightarrow \mathcal{Q}$, define

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Theorem. (L.S., RAMiCS 2018) Suppose $\mathcal{Q}$ sastifies mix:

$$x \otimes y \leq x \oplus y\,, \quad \text{for each } x, y \in Q.$$

If every interval of X is finite or if $\mathcal{Q}$ is complete, then the set of linear orders $f : X_s^2 \longrightarrow \mathcal{Q}$ is a lattice.

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Hint. The least closed $f : \mathcal{I}_X \longrightarrow \mathcal{Q}$ above $g : \mathcal{I}_X \longrightarrow \mathcal{Q}$ is given by

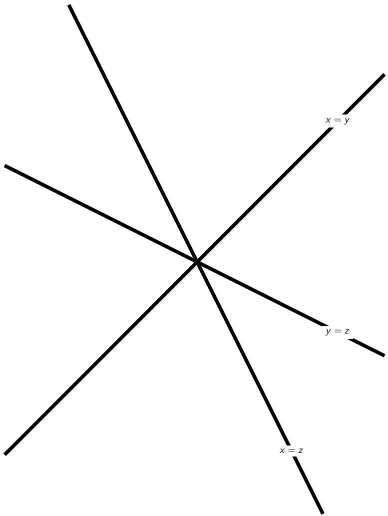
$$f(x,y) = \bigvee_{x < z_1 < \ldots < z_\ell < y} g(x, z_1) \otimes \ldots \otimes g(z_\ell, y)$$

and, moreover, if g is open, then its closure is open as well.

Various weak orders and lattices

- Order on permutations: weak Bruhat order.
- Order on pseudo-permutations: weak facial order.
- Order on multidimensional discrete paths: weak order on multipermutations (LTSTA 2016).
- Order on multidimensional continuous paths: continuous weak order (RAMICS 2018, JPAA 2021).

The geometry of the weak and facial orders



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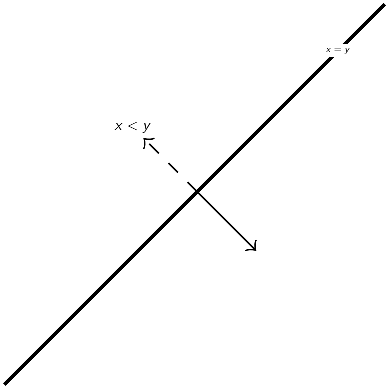

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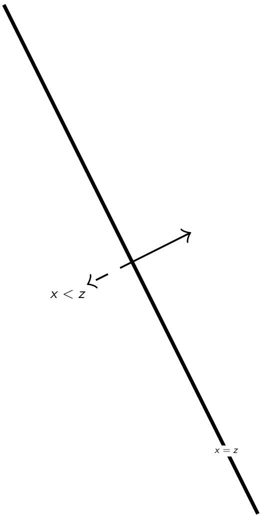

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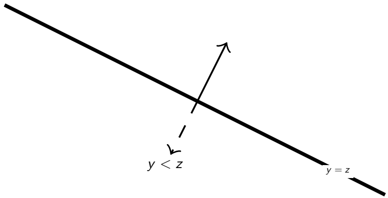

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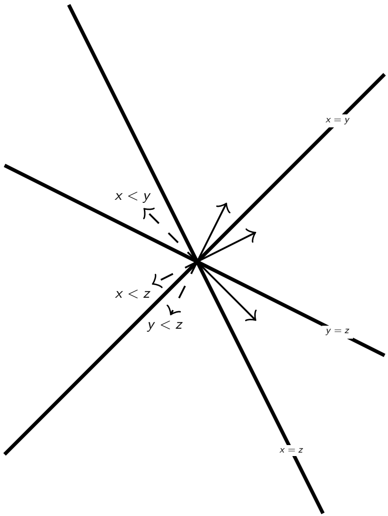

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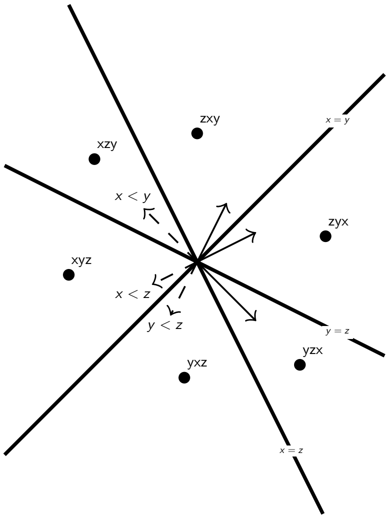

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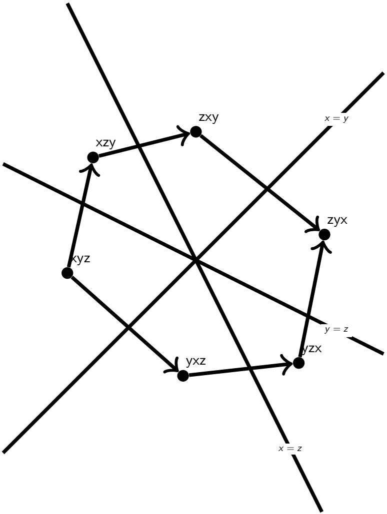

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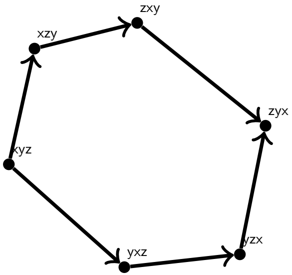

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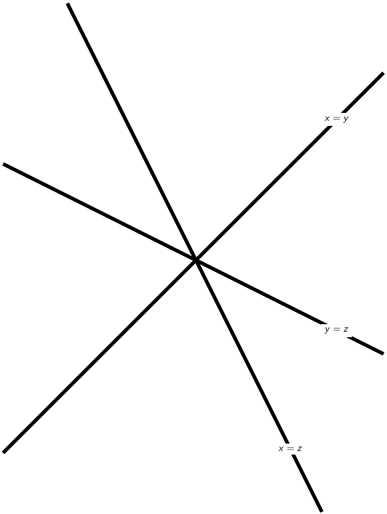

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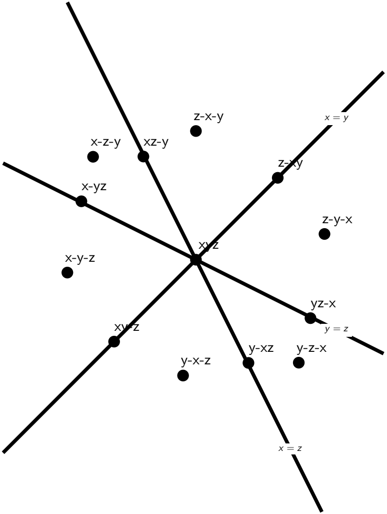

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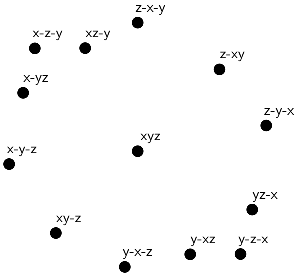

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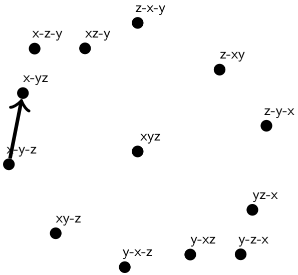

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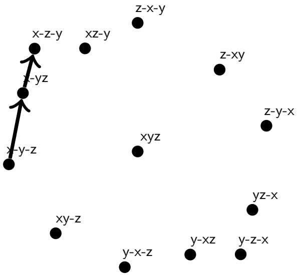

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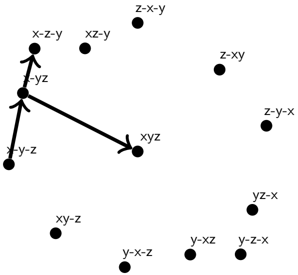

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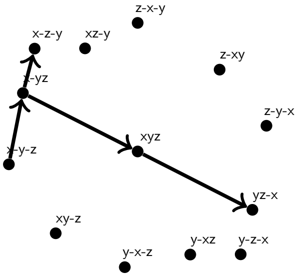

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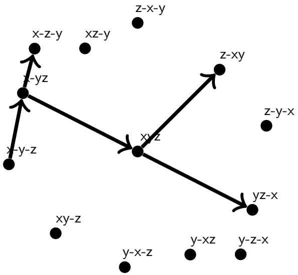

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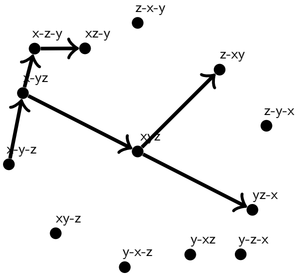

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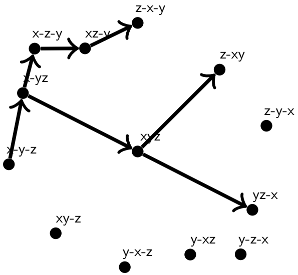

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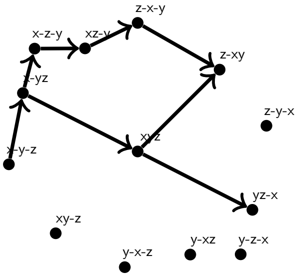

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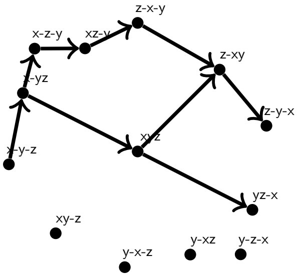

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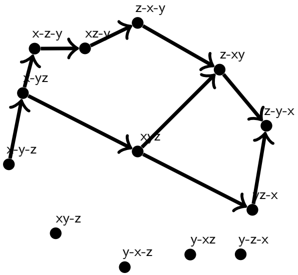

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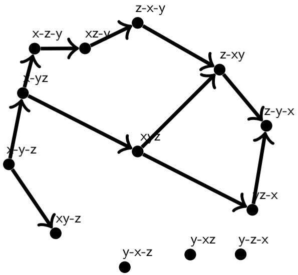

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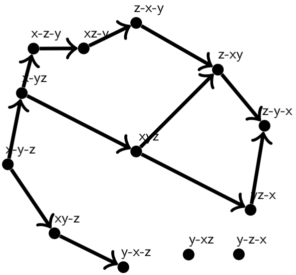

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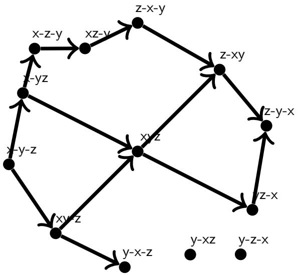

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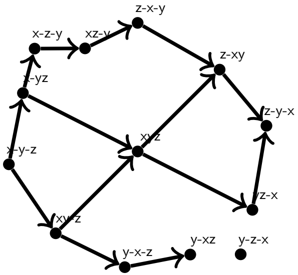

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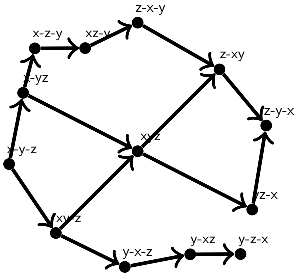

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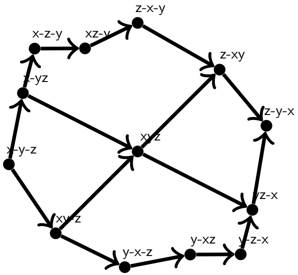

Questions and exercices

- Are linear orders valued in $\mathcal{Q}$ relevant (and for who)?
- Why are they so geometric?
- Exercise/case study :
  take your preferred (class of) involutive quantale(s) and characterise linear orders valued in it and their ordering.

  Here with the class of **Sugihara monoids**.

Plan

Linear orders valued in Girard quantales

Linear orders valued in Sugihara monoids

Sugihara monoids

Proposition. (Folklore ?) For $(C, (\cdot)^\star)$ an autodual chain, there is a unique idempotent semigroup structure $\otimes$ such that $(C, \leq, \otimes, (\cdot)^\star)$ is a Girard quantale.

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For the chain $S_{2K+1} := \{ -K, \ldots, -1, 0, 1, \ldots, K \}$ this is

$$x \otimes y = \begin{cases} x\,, & |x| > |y|\,, \\ y\,, & |x| < |y|\,, \\ -|x|\,, & |x| = |y|\,. \end{cases} \qquad x \oplus y = \begin{cases} x\,, & |x| > |y|\,, \\ y\,, & |x| < |y|\,, \\ |x|\,, & |x| = |y|\,. \end{cases}$$

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Linear orders valued in $S_3 = \mathbf{3}$ are pseudo-permutations
and the ordering is the weak facial ordering.

- **So what are linear orders valued in S_{2K+1}, for K arbitrary ?**

The combinatorial model: permutations with walls

- A permutation of $[n]$ is divided into blocks by walls.
- Letters in the same block are in increasing order.
- Each wall has an height in $\{1, \ldots, K\}$.

Example:

$$2 \;|^2\; 13 \;|^1\; 4$$

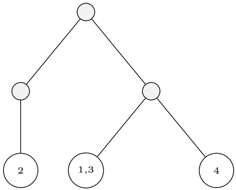
From permutations with walls to trees

Lemma. Permutations with walls bijectively correspond to rooted plane trees such that

- each branch has length K,
- the frontier is a pseudo-permutation (linear preorder).

E.g.

$2 \,|^2\, 13 \,|^1\, 4 \quad \rightsquigarrow$



From trees to linear orders valued in S_{2K+1}

Theorem. Permutations with walls and these trees bijectlvely correspond to linear orders (or skew metrics) $\ell : [n]^2_s \longrightarrow S_{2K+1}$.

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Hint. Given a linear order $\ell : [n]^2_s \longrightarrow S_{2K+1}$,

$$d(x, y) := |\ell(x, y)|$$

is an ultrametric with values in $\{ 0, \dots, K \}$, thus yielding the tree. The sign yields the underlying permutation.

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For an infinite autodual chain C:

trees are replaced by functors

$$T : C^+ \longrightarrow \texttt{LinOrd}$$

with C^+ the positive cone of C satisfying some properties.

These functors bijectively correspond to linear orders $X^2_s \longrightarrow C$.

Enumerative recreations

Proposition. Let $f(n,K)$ be the number of linear orders $\ell : [n]^2_s \longrightarrow S_{2K+1}$.

We have

$$f(n,K) = \sum_{i=1}^{n} i! \left\{ {n \atop i} \right\} K^{i-1}$$

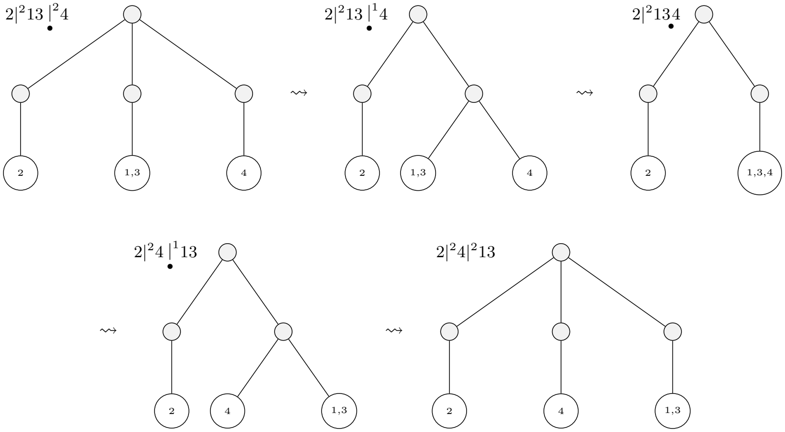
$$f(n,K) = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle K^{i} (K+1)^{n-1-i}$$

but also (Gill 1999) :

$$f(1,K) = 1, \qquad f(n+1,K) = 1 + K \sum_{i=1,\ldots,n} f(i,K)\,.$$

Characterization of covers: destroying and building walls

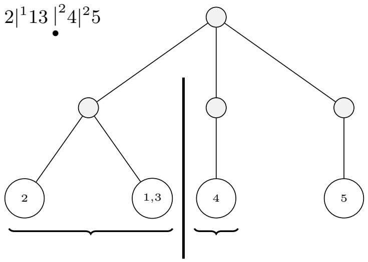
Theorem. The lattice of linear orders $\ell : \longrightarrow S_{2K+1}$ is characterized via its Hasse diagram, as legal moves (transformations) on trees/permutations with walls.



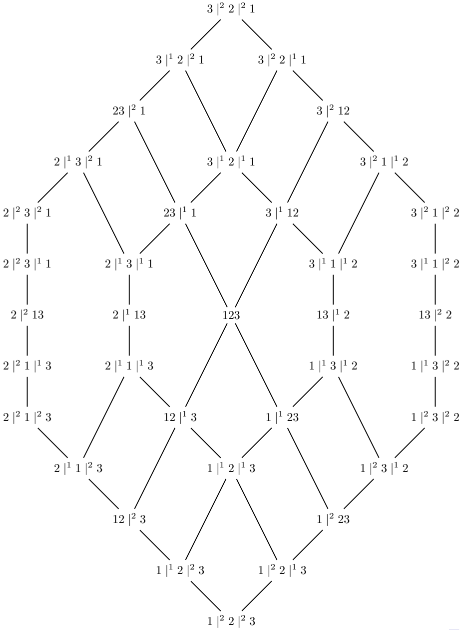
Enabled moves

- Destructing a wall is possible iff the letters on its left scope are less than those in the right scope.
- Building a wall is possible iff the letters on its left scope are greater than those in its right scope.

Left and right scopes of a wall:



Just for fun


$3 |^2 2 |^2 1$
$3 |^1 2 |^2 1$
$3 |^2 2 |^1 1$
$23 |^2 1$
$3 |^2 12$
$2 |^1 3 |^2 1$
$3 |^1 2 |^1 1$
$3 |^2 1 |^1 2$
$2 |^2 3 |^2 1$
$23 |^1 1$
$3 |^1 12$
$3 |^2 1 |^2 2$
$2 |^2 3 |^1 1$
$2 |^1 3 |^1 1$
$3 |^1 1 |^1 2$
$3 |^1 1 |^2 2$
$2 |^2 13$
$2 |^1 13$
123
$13 |^1 2$
$13 |^2 2$
$2 |^2 1 |^1 3$
$2 |^1 1 |^1 3$
$1 |^1 3 |^1 2$
$1 |^1 3 |^2 2$
$2 |^2 1 |^2 3$
$12 |^1 3$
$1 |^1 23$
$1 |^2 3 |^2 2$
$2 |^1 1 |^2 3$
$1 |^1 2 |^1 3$
$1 |^2 3 |^1 2$
$12 |^2 3$
$1 |^2 23$
$1 |^1 2 |^2 3$
$1 |^2 2 |^1 3$
$1 |^2 2 |^2 3$

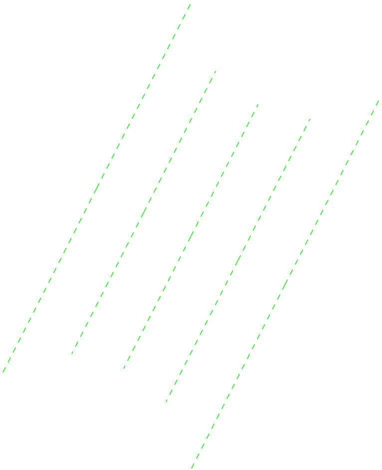
S_{2k+1} linear orders and affine braid arrangements

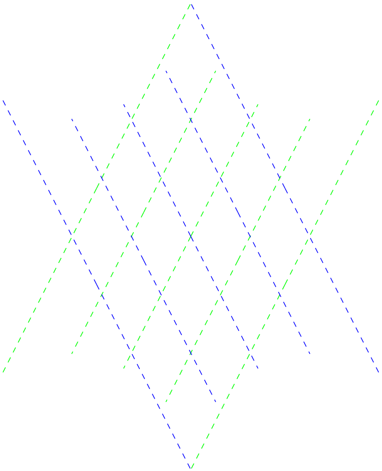
[Gill 1999] observes that permutations with walls are in bijection with "maximal" elements of the intersection poset of the affine braid arrangement

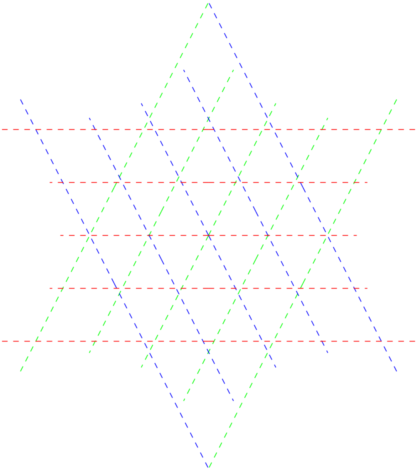
$$\mathcal{B}_{n,K} = \{ H_{i,j,k} \mid 1 \leq i < j \leq n, -K \leq k \leq K \},$$

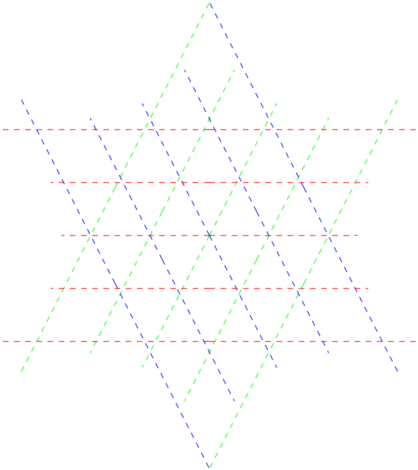
with

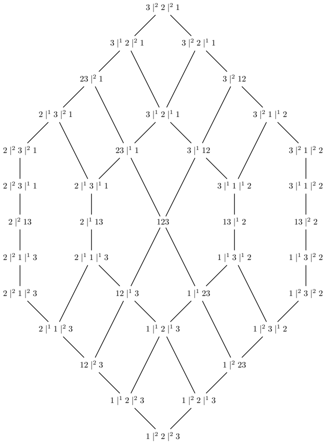
$$H_{i,j,k} = \{ \vec{x} \mid x_i = x_j + k \}.$$

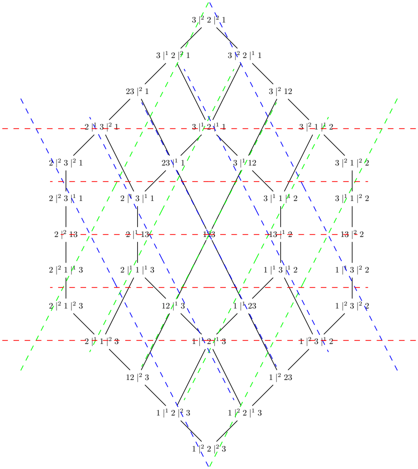









3 $|^2$ 2 $|^2$ 1
3 $|^1$ 2 $|^2$ 1
3 $|^2$ 2 $|^1$ 1
23 $|^2$ 1
3 $|^2$ 12
2 $|^1$ 3 $|^2$ 1
3 $|^1$ 2 $|^1$ 1
3 $|^2$ 1 $|^1$ 2
2 $|^2$ 3 $|^2$ 1
23 $|^1$ 1
3 $|^1$ 12
3 $|^2$ 1 $|^2$ 2
2 $|^2$ 3 $|^1$ 1
2 $|^1$ 3 $|^1$ 1
3 $|^1$ 1 $|^1$ 2
3 $|^1$ 1 $|^2$ 2
2 $|^2$ 13
2 $|^1$ 13
123
13 $|^1$ 2
13 $|^2$ 2
2 $|^2$ 1 $|^1$ 3
2 $|^1$ 1 $|^1$ 3
1 $|^1$ 3 $|^1$ 2
1 $|^1$ 3 $|^2$ 2
2 $|^2$ 1 $|^2$ 3
12 $|^1$ 3
1 $|^1$ 23
1 $|^2$ 3 $|^2$ 2
2 $|^1$ 1 $|^2$ 3
1 $|^1$ 2 $|^1$ 3
1 $|^2$ 3 $|^1$ 2
12 $|^2$ 3
1 $|^2$ 23
1 $|^1$ 2 $|^2$ 3
1 $|^2$ 2 $|^1$ 3
1 $|^2$ 2 $|^2$ 3


3 |² 2 |² 1
3 |¹ 2 |² 1
3 |² 2 |¹ 1
23 |² 1
3 |² 12
2 |¹ 3 |² 1
3 |¹ 2 |¹ 1
3 |² 1 |¹ 2
2 |² 3 |² 1
23 |¹ 1
3 |¹ 12
3 |² 1 |² 2
2 |² 3 |¹ 1
2 |¹ 3 |¹ 1
3 |¹ 1 |¹ 2
3 |¹ 1 |² 2
2 |² 13
2 |¹ 13
123
13 |¹ 2
13 |² 2
2 |² 1 |¹ 3
2 |¹ 1 |¹ 3
1 |¹ 3 |¹ 2
1 |¹ 3 |² 2
2 |² 1 |² 3
12 |¹ 3
1 |¹ 23
1 |² 3 |² 2
2 |¹ 1 |² 3
1 |¹ 2 |¹ 3
1 |² 3 |¹ 2
12 |² 3
1 |² 23
1 |¹ 2 |² 3
1 |² 2 |¹ 3
1 |² 2 |² 3

Conclusions, future directions

- Characterisation of the ordering on linear orders

$$\ell : [n]^2_s \longrightarrow S_{2K+1}$$

  via a combinatorial description of its Hasse diagram.

- More enumerative results:
  - Number of join-irreducible elements?
  - S_{2K+1}-Eulerian numbers?

- Further connections with affine braid arrangements:
  - Geometric interpretation of moves?

- These are type A linear orders valued in $\mathcal{Q}$.
  Are there type B and D linear orders valued in $\mathcal{Q}$?

- Linear orders valued in MV-algebras?

- Back to Sugihara monoids:
  link with fixedpoint theory and parity games (generalizing Büchi conditions)?

Thank you !