Time Warps, from Algebra to Algorithms

Simon Santschi

Mathematical Institute
University of Bern

Joint work with Sam van Gool, Adrien Guatto, and George Metcalfe

RAMICS 2021

November 3, 2021

Time Warps

Recall that $\omega^+ = \omega \cup \{\omega\}$. We call a map $f\colon \omega^+ \to \omega^+$ a **time warp** if it is join-preserving (for all $S \subseteq \omega^+$, $f(\bigvee S) = \bigvee f[S]$).

Time Warps

Recall that $\omega^+ = \omega \cup \{\omega\}$. We call a map $f\colon \omega^+ \to \omega^+$ a **time warp** if it is join-preserving (for all $S \subseteq \omega^+$, $f(\bigvee S) = \bigvee f[S]$).

The set of time warps $\mathscr{W}$ gives rise to a **bounded residuated lattice** $\mathbf{W} = \langle \mathscr{W}, \wedge, \vee, \circ, \backslash, /, id, \bot, \top \rangle$, i.e.,

Time Warps

Recall that $\omega^+ = \omega \cup \{\omega\}$. We call a map $f\colon \omega^+ \to \omega^+$ a **time warp** if it is join-preserving (for all $S \subseteq \omega^+$, $f(\bigvee S) = \bigvee f[S]$).

The set of time warps $\mathscr{W}$ gives rise to a **bounded residuated lattice** $\mathbf{W} = \langle \mathscr{W}, \wedge, \vee, \circ, \backslash, /, id, \bot, \top \rangle$, i.e.,

- $\langle \mathscr{W}, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $\wedge$ and $\vee$ defined point-wise (e.g., $(f \wedge g)(p) = \max\{f(p), g(p)\}$), $\bot(p) = 0$ for all $p \in \omega^+$, and $\top(p) = \omega$ for all $p \in \omega^+ \setminus \{0\}$.

Time Warps

Recall that $\omega^+ = \omega \cup \{\omega\}$. We call a map $f\colon \omega^+ \to \omega^+$ a **time warp** if it is join-preserving (for all $S \subseteq \omega^+$, $f(\bigvee S) = \bigvee f[S]$).

The set of time warps $\mathscr{W}$ gives rise to a **bounded residuated lattice** $\mathbf{W} = \langle \mathscr{W}, \wedge, \vee, \circ, \backslash, /, id, \bot, \top \rangle$, i.e.,

- $\langle \mathscr{W}, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $\wedge$ and $\vee$ defined point-wise (e.g., $(f \wedge g)(p) = \max\{f(p), g(p)\}$), $\bot(p) = 0$ for all $p \in \omega^+$, and $\top(p) = \omega$ for all $p \in \omega^+ \setminus \{0\}$.
- $\langle \mathscr{W}, \circ, id \rangle$ is a monoid with $(f \circ g)(p) = f(g(p))$ and id the identity.

Time Warps

Recall that $\omega^+ = \omega \cup \{\omega\}$. We call a map $f\colon \omega^+ \to \omega^+$ a **time warp** if it is join-preserving (for all $S \subseteq \omega^+$, $f(\bigvee S) = \bigvee f[S]$).

The set of time warps $\mathscr{W}$ gives rise to a **bounded residuated lattice** $\mathbf{W} = \langle \mathscr{W}, \wedge, \vee, \circ, \backslash, /, id, \bot, \top \rangle$, i.e.,

- $\langle \mathscr{W}, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $\wedge$ and $\vee$ defined point-wise (e.g., $(f \wedge g)(p) = \max\{f(p), g(p)\}$), $\bot(p) = 0$ for all $p \in \omega^+$, and $\top(p) = \omega$ for all $p \in \omega^+ \setminus \{0\}$.
- $\langle \mathscr{W}, \circ, id \rangle$ is a monoid with $(f \circ g)(p) = f(g(p))$ and id the identity.
- For all $f, g, h \in \mathscr{W}$

$$f \circ g \le h \iff g \le f \backslash h \iff f \le h/g, \qquad \text{(residuation)}$$

where $f \backslash g = \bigvee \{h \in \mathscr{W} \mid f \circ h \le g\}$, $g/f = \bigvee \{h \in \mathscr{W} \mid h \circ f \le g\}$.

Time Warps

Recall that $\omega^+ = \omega \cup \{\omega\}$. We call a map $f : \omega^+ \to \omega^+$ a **time warp** if it is join-preserving (for all $S \subseteq \omega^+$, $f(\bigvee S) = \bigvee f[S]$).

The set of time warps $\mathscr{W}$ gives rise to a **bounded residuated lattice** $\mathbf{W} = \langle \mathscr{W}, \wedge, \vee, \circ, \backslash, /, id, \bot, \top \rangle$, i.e.,

- $\langle \mathscr{W}, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $\wedge$ and $\vee$ defined point-wise (e.g., $(f \wedge g)(p) = \max\{f(p), g(p)\}$), $\bot(p) = 0$ for all $p \in \omega^+$, and $\top(p) = \omega$ for all $p \in \omega^+ \setminus \{0\}$.
- $\langle \mathscr{W}, \circ, id \rangle$ is a monoid with $(f \circ g)(p) = f(g(p))$ and id the identity.
- For all $f, g, h \in \mathscr{W}$

$$f \circ g \leq h \iff g \leq f \backslash h \iff f \leq h/g, \qquad \text{(residuation)}$$

where $f \backslash g = \bigvee \{h \in \mathscr{W} \mid f \circ h \leq g\}$, $g/f = \bigvee \{h \in \mathscr{W} \mid h \circ f \leq g\}$.

We call $\mathbf{W}$ the **time warp algebra**.

Properties of the Time Warp Algebra

Properties

1. *A map* $f\colon \omega^+ \to \omega^+$ *is a time warp if and only if it is order-preserving and satisfies* $f(0) = 0$ *and* $f(\omega) = \bigvee\{f(n) \mid n \in \omega\}$*.*

Properties of the Time Warp Algebra

Properties

1. *A map* $f : \omega^+ \to \omega^+$ *is a time warp if and only if it is order-preserving and satisfies* $f(0) = 0$ *and* $f(\omega) = \bigvee\{f(n) \mid n \in \omega\}$.
2. $\langle \mathscr{W}, \wedge, \vee \rangle$ *is a complete distributive lattice.*

Properties of the Time Warp Algebra

Properties

1. *A map* $f\colon \omega^+ \to \omega^+$ *is a time warp if and only if it is order-preserving and satisfies* $f(0) = 0$ *and* $f(\omega) = \bigvee\{f(n) \mid n \in \omega\}$.
2. $\langle \mathscr{W}, \wedge, \vee \rangle$ *is a complete distributive lattice.*
3. *For all* $f, g_1, g_2, h \in \mathscr{W}$,

$$f(g_1 \vee g_2)h = fg_1h \vee fg_2h \text{ and } f(g_1 \wedge g_2)h = fg_1h \wedge fg_2h.$$

Motivation

Why do we care about the time warp algebra?

Motivation

Why do we care about the time warp algebra?

- Computer Science: Time warps can be used as gradings for graded modalities in type systems (Guatto 2018). This is also where the name 'time warp' comes from.

Motivation

Why do we care about the time warp algebra?

- Computer Science: Time warps can be used as gradings for graded modalities in type systems (Guatto 2018). This is also where the name 'time warp' comes from.
- Universal Algebra: Endomorphism algebras are natural to consider, e.g., automorphism ℓ-groups of chains in the theory of lattice-ordered groups, or, more closely related, quantales of sup-preserving functions on complete lattices (see e.g., Santocanale 2020).

Motivation

Why do we care about the time warp algebra?

- Computer Science: Time warps can be used as gradings for graded modalities in type systems (Guatto 2018). This is also where the name 'time warp' comes from.
- Universal Algebra: Endomorphism algebras are natural to consider, e.g., automorphism ℓ-groups of chains in the theory of lattice-ordered groups, or, more closely related, quantales of sup-preserving functions on complete lattices (see e.g., Santocanale 2020).

For potential real-world applications of time warps as graded modalities it is important to have a decidable equational theory, i.e., an algorithm to decide which equations hold in the time warp algebra.

Motivation

Why do we care about the time warp algebra?

- Computer Science: Time warps can be used as gradings for graded modalities in type systems (Guatto 2018). This is also where the name 'time warp' comes from.
- Universal Algebra: Endomorphism algebras are natural to consider, e.g., automorphism ℓ-groups of chains in the theory of lattice-ordered groups, or, more closely related, quantales of sup-preserving functions on complete lattices (see e.g., Santocanale 2020).

For potential real-world applications of time warps as graded modalities it is important to have a decidable equational theory, i.e., an algorithm to decide which equations hold in the time warp algebra.

Main Theorem

The equational theory of the time warp algebra $\mathbf{W}$ *is decidable.*

Notation

We fix a countably infinite set of variables Var and the term algebra $\mathbf{T}(\mathrm{Var})$ over the language $\{\wedge, \vee, \circ, \backslash, /, id, \bot, \top\}$ of type $(2, 2, 2, 2, 0, 0, 0)$.

Notation

We fix a countably infinite set of variables Var and the term algebra $\mathbf{T}(\mathrm{Var})$ over the language $\{\wedge, \vee, \circ, \backslash, /, id, \bot, \top\}$ of type $(2, 2, 2, 2, 0, 0, 0)$.

We call elements t of $\mathbf{T}(\mathrm{Var})$ **time warp terms** and denote by $s \leq t$ the equation $s \wedge t \approx s$.

Notation

We fix a countably infinite set of variables Var and the term algebra $\mathbf{T}(\mathrm{Var})$ over the language $\{\wedge, \vee, \circ, \backslash, /, id, \bot, \top\}$ of type $(2, 2, 2, 2, 0, 0, 0)$.

We call elements t of $\mathbf{T}(\mathrm{Var})$ **time warp terms** and denote by $s \leq t$ the equation $s \wedge t \approx s$.

Then we have $\mathbf{W} \models s \approx t$ if and only if $\mathbf{W} \models s \leq t$ and $\mathbf{W} \models t \leq s$, and, by residuation, $\mathbf{W} \models s \leq t$ if and only if $\mathbf{W} \models id \leq s\backslash t$.

Notation

We fix a countably infinite set of variables Var and the term algebra $\mathbf{T}(\mathrm{Var})$ over the language $\{\wedge, \vee, \circ, \backslash, /, id, \bot, \top\}$ of type $(2, 2, 2, 2, 0, 0, 0)$.

We call elements t of $\mathbf{T}(\mathrm{Var})$ **time warp terms** and denote by $s \leq t$ the equation $s \wedge t \approx s$.

Then we have $\mathbf{W} \models s \approx t$ if and only if $\mathbf{W} \models s \leq t$ and $\mathbf{W} \models t \leq s$, and, by residuation, $\mathbf{W} \models s \leq t$ if and only if $\mathbf{W} \models id \leq s \backslash t$.

Therefore, to show that the equational theory of $\mathbf{W}$ is decidable it is enough to show that for every time warp term t it is decidable whether $\mathbf{W} \models id \leq t$ holds or not.

Overview of the Proof

We prove the main theorem by describing an algorithm with the following behaviour:

Overview of the Proof

We prove the main theorem by describing an algorithm with the following behaviour:

Input. A time warp term t in the variables $x_1, \ldots, x_k$.

Output. If $\mathbf{W} \models id \leq t$, the algorithm returns '`Valid`';
if $\mathbf{W} \not\models id \leq t$, the algorithm returns '`Invalid at` $(\hat{f}_1, \ldots, \hat{f}_k, p)$' for some $p \in \omega^+$ and finite descriptions $\hat{f}_1, \ldots, \hat{f}_k$ of time warps $f_1, \ldots, f_k$ such that $[\![t]\!](p) < p$, where $[\![t]\!]$ is the time warp obtained from t by mapping each x_i to f_i.

Overview of the Proof

The proof of the main theorem can be divided into three parts:

Overview of the Proof

The proof of the main theorem can be divided into three parts:

1. Step 1. We prove that time warp terms can be 'brought' into a normal form.

Overview of the Proof

The proof of the main theorem can be divided into three parts:

1. Step 1. We prove that time warp terms can be ‘brought’ into a normal form.
2. Step 2. We give a finitary characterization of ‘potential counterexamples’ via ‘diagrams’[1].

[1] The name ‘diagram’ recalls a similar concept used to prove the decidability of the equational theory of ℓ-groups (Holland and McCleary 1979).

Overview of the Proof

The proof of the main theorem can be divided into three parts:

1. Step 1. We prove that time warp terms can be 'brought' into a normal form.
2. Step 2. We give a finitary characterization of 'potential counterexamples' via 'diagrams'[1].
3. Step 3. We encode the existence of a 'diagram' as a first-order satisfiability problem over $\langle \mathbb{N}, \leq^{\mathbb{N}} \rangle$.

[1] The name 'diagram' recalls a similar concept used to prove the decidability of the equational theory of ℓ-groups (Holland and McCleary 1979).

Step 1. A Normal Form for Time Warps

For a time warp f we define

$$f^{\ell} := id/f, \quad f^{\mathsf{r}} := f\backslash id, \quad \text{and} \quad f^{\mathsf{o}} := \top\backslash f.$$

and we call terms constructed using only the operations $\circ, id, \perp$ and the defined operations $t^{\ell} = id/t$, $t^{\mathsf{r}} = t\backslash id$, and $t^{\mathsf{o}} = \top\backslash t$ **basic terms**.

Step 1. A Normal Form for Time Warps

For a time warp f we define

$$f^{\ell} := id/f, \quad f^{\mathsf{r}} := f\backslash id, \quad \text{and} \quad f^{\mathsf{o}} := \top\backslash f.$$

and we call terms constructed using only the operations $\circ, id, \bot$ and the defined operations $t^{\ell} = id/t$, $t^{\mathsf{r}} = t\backslash id$, and $t^{\mathsf{o}} = \top\backslash t$ **basic terms**.

One can show that join and meet 'distribute' over the residuals and that for any time warps f, g,

$$f\backslash g = f^{\mathsf{r}} g \vee (\top f)^{\mathsf{r}} \vee g^{\mathsf{o}} \quad \text{and} \quad g/f = gf^{\ell} \vee (f^{\ell})^{\mathsf{o}}$$

Step 1. A Normal Form for Time Warps

For a time warp f we define

$$f^{\ell} := id/f, \quad f^{\mathsf{r}} := f\backslash id, \quad \text{and} \quad f^{\mathsf{o}} := \top\backslash f.$$

and we call terms constructed using only the operations $\circ, id, \bot$ and the defined operations $t^{\ell} = id/t$, $t^{\mathsf{r}} = t\backslash id$, and $t^{\mathsf{o}} = \top\backslash t$ **basic terms**.

One can show that join and meet 'distribute' over the residuals and that for any time warps f, g,

$$f\backslash g = f^{\mathsf{r}}g \vee (\top f)^{\mathsf{r}} \vee g^{\mathsf{o}} \quad \text{and} \quad g/f = gf^{\ell} \vee (f^{\ell})^{\mathsf{o}}$$

Theorem

There is an effective procedure that given any time warp term t, produces positive integers $m, n_1, \ldots, n_m$ and a set of basic time warp terms $\{t_{i,j} \mid 1 \leq i \leq m;\ 1 \leq j \leq n_i\}$ satisfying $\mathbf{W} \models t \approx \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n_i} t_{i,j}$.

Step 1. A Normal Form for Time Warps

In universal algebra terms the normal form theorem states that the time warp algebra is term equivalent to the algebra $\langle \mathscr{W}, \wedge, \vee, \circ, {}^{\mathsf{r}}, {}^{\ell}, {}^{\mathsf{o}}, id, \bot \rangle$, so as a direct consequence we get:

Corollary

The equational theory of $\mathbf{W}$ *is decidable if, and only if, there exists an effective procedure that decides for any finite non-empty set of basic time warp terms* $\{t_1, \ldots, t_n\}$ *if* $\mathbf{W} \models id \leq t_1 \vee \cdots \vee t_n$.

Step 1. A Normal Form for Time Warps

In universal algebra terms the normal form theorem states that the time warp algebra is term equivalent to the algebra $\langle \mathscr{W}, \wedge, \vee, \circ, {}^{\mathsf{r}}, {}^{\ell}, {}^{\mathsf{o}}, id, \bot \rangle$, so as a direct consequence we get:

Corollary

The equational theory of $\mathbf{W}$ is decidable if, and only if, there exists an effective procedure that decides for any finite non-empty set of basic time warp terms $\{t_1, \ldots, t_n\}$ if $\mathbf{W} \models id \leq t_1 \vee \cdots \vee t_n$.

Accordingly in the following we will consider joins of basic terms. We extend a valuation $\theta \colon \mathrm{Var} \to \mathscr{W}$ inductively to basic terms

$$[\![x]\!]_\theta := \theta(x), \quad [\![id]\!]_\theta := id, \quad [\![\bot]\!]_\theta := \bot,$$

$$[\![tu]\!]_\theta := [\![t]\!]_\theta [\![u]\!]_\theta, \quad [\![t^\star]\!]_\theta := [\![t]\!]_\theta^\star \text{ for } \star \in \{\mathsf{o}, \ell, \mathsf{r}\}.$$

Step 2. Samples

Step 2. Samples

Goal: Find for $id \leq t$ a finitary way to describe the relevant information of a counterexample $\theta \colon \mathrm{Var} \to \mathscr{W}$, $p \in \omega$, such that $[\![t]\!]_\theta(p) < p$.

Step 2. Samples

Goal: Find for $id \leq t$ a finitary way to describe the relevant information of a counterexample $\theta \colon \mathrm{Var} \to \mathscr{W}$, $p \in \omega$, such that $[\![t]\!]_\theta(p) < p$.

The idea is to define syntactic objects which we want to associate with points in ω^+.

Step 2. Samples

Goal: Find for $id \leq t$ a finitary way to describe the relevant information of a counterexample $\theta\colon \mathrm{Var} \to \mathscr{W}$, $p \in \omega$, such that $[\![t]\!]_\theta(p) < p$.

The idea is to define syntactic objects which we want to associate with points in ω^+.

We fix a countably infinite set $\mathscr{I}_V$ of **time variables** which we denote by κ, κ', etc. and we define a **sample** to be an object belonging to the following grammar (where t is any basic term)

$$\mathscr{I} \ni \alpha ::= \kappa \mid t[\alpha] \mid \mathsf{s}(\alpha) \mid \mathsf{p}(\alpha) \mid \mathsf{last}(t).$$

Samples are purely syntactic, but the notation already suggests the intended meaning.

Step 2. Saturated Sample Sets

We say that a sample set Δ is **saturated** if whenever $\alpha \in \Delta$ and $\alpha \rightsquigarrow \beta$, then $\beta \in \Delta$, where $\rightsquigarrow$ is the relation between samples defined by

$$
\begin{aligned}
t[\alpha] &\rightsquigarrow \alpha & t^{\mathsf{o}}[\alpha] &\rightsquigarrow t[\alpha] \\
\mathsf{s}(\alpha) &\rightsquigarrow \alpha & t^{\mathsf{r}}[\alpha] &\rightsquigarrow t[t^{\mathsf{r}}[\alpha]], t[\mathsf{s}(t^{\mathsf{r}}[\alpha])] \\
\mathsf{p}(\alpha) &\rightsquigarrow \alpha & t^{\ell}[\alpha] &\rightsquigarrow t[t^{\ell}[\alpha]], t[\mathsf{p}(t^{\ell}[\alpha])] \\
tu[\alpha] &\rightsquigarrow t[u[\alpha]] & t[\alpha] &\rightsquigarrow t[\mathsf{last}(t)].
\end{aligned}
$$

Step 2. Saturated Sample Sets

We say that a sample set Δ is **saturated** if whenever $\alpha \in \Delta$ and $\alpha \rightsquigarrow \beta$, then $\beta \in \Delta$, where $\rightsquigarrow$ is the relation between samples defined by

$$\begin{aligned} t[\alpha] &\rightsquigarrow \alpha & t^{\circ}[\alpha] &\rightsquigarrow t[\alpha] \\ \mathsf{s}(\alpha) &\rightsquigarrow \alpha & t^{\mathsf{r}}[\alpha] &\rightsquigarrow t[t^{\mathsf{r}}[\alpha]], t[\mathsf{s}(t^{\mathsf{r}}[\alpha])] \\ \mathsf{p}(\alpha) &\rightsquigarrow \alpha & t^{\ell}[\alpha] &\rightsquigarrow t[t^{\ell}[\alpha]], t[\mathsf{p}(t^{\ell}[\alpha])] \\ tu[\alpha] &\rightsquigarrow t[u[\alpha]] & t[\alpha] &\rightsquigarrow t[\mathsf{last}(t)]. \end{aligned}$$

The **saturation** of a sample set Δ is defined as

$$\Delta^{\rightsquigarrow} := \{\beta \mid \exists \alpha \in \Delta, \alpha \rightsquigarrow^{*} \beta\},$$

where $\rightsquigarrow^{*}$ denotes the reflexive transitive closure of $\rightsquigarrow$.

Step 2. Saturated Sample Sets

We say that a sample set Δ is **saturated** if whenever $\alpha \in \Delta$ and $\alpha \rightsquigarrow \beta$, then $\beta \in \Delta$, where $\rightsquigarrow$ is the relation between samples defined by

$$
\begin{array}{ll}
t[\alpha] \rightsquigarrow \alpha & t^{\mathsf{o}}[\alpha] \rightsquigarrow t[\alpha] \\
\mathsf{s}(\alpha) \rightsquigarrow \alpha & t^{\mathsf{r}}[\alpha] \rightsquigarrow t[t^{\mathsf{r}}[\alpha]], t[\mathsf{s}(t^{\mathsf{r}}[\alpha])] \\
\mathsf{p}(\alpha) \rightsquigarrow \alpha & t^{\ell}[\alpha] \rightsquigarrow t[t^{\ell}[\alpha]], t[\mathsf{p}(t^{\ell}[\alpha])] \\
tu[\alpha] \rightsquigarrow t[u[\alpha]] & t[\alpha] \rightsquigarrow t[\mathsf{last}(t)].
\end{array}
$$

The **saturation** of a sample set Δ is defined as

$$\Delta^{\rightsquigarrow} := \{\beta \mid \exists \alpha \in \Delta, \alpha \rightsquigarrow^{*} \beta\},$$

where $\rightsquigarrow^{*}$ denotes the reflexive transitive closure of $\rightsquigarrow$.

Lemma

If Δ is a finite sample set, then its saturation $\Delta^{\rightsquigarrow}$ is also finite.

Step 2. Diagrams

Let us fix a saturated sample set Δ. A **Δ-prediagram** is a map $\delta\colon \Delta \to \omega^+$.

Step 2. Diagrams

Let us fix a saturated sample set Δ. A **Δ-prediagram** is a map $\delta\colon \Delta \to \omega^+$.

We call a Δ-prediagram a **Δ-diagram** if it satisfies a number of conditions. The first four conditions are

$$\forall t[\alpha], t[\beta] \in \Delta,\ \delta(\alpha) \leq \delta(\beta) \Rightarrow \delta(t[\alpha]) \leq \delta(t[\beta]) \tag{1}$$

$$\forall t[\alpha] \in \Delta,\ \delta(\alpha) = 0 \Rightarrow \delta(t[\alpha]) = 0 \tag{2}$$

$$\forall \mathsf{p}(\alpha) \in \Delta,\ \delta(\mathsf{p}(\alpha)) = \delta(\alpha) \ominus 1 \tag{3}$$

$$\forall \mathsf{s}(\alpha) \in \Delta,\ \delta(\mathsf{s}(\alpha)) = \delta(\alpha) \oplus 1 \tag{4}$$

Step 2. Diagrams

Let us fix a saturated sample set Δ. A **Δ-prediagram** is a map $\delta\colon \Delta \to \omega^+$.

We call a Δ-prediagram a **Δ-diagram** if it satisfies a number of conditions. The first four conditions are

$$\forall t[\alpha], t[\beta] \in \Delta,\ \delta(\alpha) \leq \delta(\beta) \Rightarrow \delta(t[\alpha]) \leq \delta(t[\beta]) \tag{1}$$

$$\forall t[\alpha] \in \Delta,\ \delta(\alpha) = 0 \Rightarrow \delta(t[\alpha]) = 0 \tag{2}$$

$$\forall \mathsf{p}(\alpha) \in \Delta,\ \delta(\mathsf{p}(\alpha)) = \delta(\alpha) \ominus 1 \tag{3}$$

$$\forall \mathsf{s}(\alpha) \in \Delta,\ \delta(\mathsf{s}(\alpha)) = \delta(\alpha) \oplus 1 \tag{4}$$

$$p \ominus 1 := \begin{cases} p - 1 & \text{if } p \in \omega \setminus \{0\} \\ p & \text{if } p \in \{0, \omega\} \end{cases}, \qquad p \oplus 1 := \begin{cases} p + 1 & \text{if } p \in \omega \\ p & \text{if } p = \omega \end{cases}$$

Step 2. Diagrams

Let us fix a saturated sample set Δ. A **Δ-prediagram** is a map $\delta\colon \Delta \to \omega^+$.

We call a Δ-prediagram a **Δ-diagram** if it satisfies a number of conditions. The first four conditions are

$$\forall t[\alpha], t[\beta] \in \Delta,\ \delta(\alpha) \leq \delta(\beta) \Rightarrow \delta(t[\alpha]) \leq \delta(t[\beta]) \tag{1}$$

$$\forall t[\alpha] \in \Delta,\ \delta(\alpha) = 0 \Rightarrow \delta(t[\alpha]) = 0 \tag{2}$$

$$\forall \mathsf{p}(\alpha) \in \Delta,\ \delta(\mathsf{p}(\alpha)) = \delta(\alpha) \ominus 1 \tag{3}$$

$$\forall \mathsf{s}(\alpha) \in \Delta,\ \delta(\mathsf{s}(\alpha)) = \delta(\alpha) \oplus 1 \tag{4}$$

There are 19 more conditions which capture how the three constants, the product, and the three residuals behave. For example condition (16) is

$$\forall t^{\mathsf{r}}[\alpha] \in \Delta,\ (0 < \delta(\alpha) < \omega \text{ and } \delta(t^{\mathsf{r}}[\alpha]) < \omega) \Rightarrow \delta(\alpha) < \delta(t[\mathsf{s}(t^{\mathsf{r}}[\alpha])])$$

Step 2. From Valuations to Diagrams

Proposition

Let T be a set of basic terms, κ a time variable, and Δ the saturation of the sample set $\{t[\kappa] \mid t \in T\}$. Then for any valuation θ and $p \in \omega^+$, there exists a Δ-diagram δ such that $\delta(\kappa) = p$ and $\delta(t[\kappa]) = [\![t]\!]_\theta(p)$ for all $t \in T$.

Step 2. From Valuations to Diagrams

Proposition

Let T be a set of basic terms, κ a time variable, and Δ the saturation of the sample set $\{t[\kappa] \mid t \in T\}$. Then for any valuation θ and $p \in \omega^+$, there exists a Δ-diagram δ such that $\delta(\kappa) = p$ and $\delta(t[\kappa]) = [\![t]\!]_\theta(p)$ for all $t \in T$.

Define the map $\delta\colon \Delta \to \omega^+$ by structural induction on the samples in Δ as follows

$$\begin{aligned}
& \delta(\kappa) && := p \\
\forall \mathsf{last}(t) \in \Delta, \quad & \delta(\mathsf{last}(t)) && := \bigwedge\{p \in \omega^+ \mid [\![t]\!]_\theta(p) = [\![t]\!]_\theta(\omega)\} \\
\forall t[\alpha] \in \Delta, \quad & \delta(t[\alpha]) && := [\![t]\!]_\theta(\delta(\alpha)) \\
\forall \mathsf{p}(\alpha) \in \Delta, \quad & \delta(\mathsf{p}(\alpha)) && := \delta(\alpha) \ominus 1 \\
\forall \mathsf{s}(\alpha) \in \Delta, \quad & \delta(\mathsf{s}(\alpha)) && := \delta(\alpha) \oplus 1.
\end{aligned}$$

Step 2. From Diagrams to Valuations

To go from diagrams to valuations we define for a Δ-diagram δ and basic term t

$$\lfloor t \rfloor_\delta \coloneqq \{(\delta(\alpha), \delta(t[\alpha])) \mid t[\alpha] \in \Delta\}.$$

Note that $\lfloor t \rfloor_\delta$ is a partial map from ω^+ to ω^+.

Step 2. From Diagrams to Valuations

To go from diagrams to valuations we define for a Δ-diagram δ and basic term t

$$\lfloor t \rfloor_\delta \coloneqq \{(\delta(\alpha), \delta(t[\alpha])) \mid t[\alpha] \in \Delta\}.$$

Note that $\lfloor t \rfloor_\delta$ is a partial map from ω^+ to ω^+.

Proposition

There is an effective procedure that produces for any finite Δ-diagram δ, an algorithmic description of a valuation θ satisfying $[\![t]\!]_\theta(\delta(\alpha)) = \lfloor t \rfloor_\delta(\delta(\alpha))$ for all $t[\alpha] \in \Delta$.

Step 2. Diagram Theorem

Summarizing the two propositions we get

Theorem (Diagram Theorem)

Let $t_1, \ldots, t_n$ *be basic terms,* κ *a time variable, and* Δ *the saturation of the sample set* $\{t_1[\kappa], \ldots, t_n[\kappa]\}$*. Then* $\mathbf{W} \not\models id \leq t_1 \vee \cdots \vee t_n$ *if, and only if, there exists a* Δ*-diagram* δ *such that* $\delta(\kappa) > \delta(t_i[\kappa])$ *for all* $i \in \{1, \ldots, n\}$*.*

Step 2. Diagram Theorem

Summarizing the two propositions we get

Theorem (Diagram Theorem)

Let $t_1, \ldots, t_n$ be basic terms, κ a time variable, and Δ the saturation of the sample set $\{t_1[\kappa], \ldots, t_n[\kappa]\}$. Then $\mathbf{W} \not\models id \leq t_1 \vee \cdots \vee t_n$ if, and only if, there exists a Δ-diagram δ such that $\delta(\kappa) > \delta(t_i[\kappa])$ for all $i \in \{1, \ldots, n\}$.

So what remains to do is to describe an algorithm to decide whether a suitable diagram exists or not.

Step 3. Translation into Logic

Fix $\Delta = \{t_1[\kappa], \ldots, t_n[\kappa]\}^{\rightsquigarrow}$. We consider the first-order (relational) signature $\tau = \{\preceq, \mathcal{S}, \mathcal{O}, \mathcal{I}\}$ of type $(2, 2, 1, 1)$ and ω^+ as a τ structure by defining

Step 3. Translation into Logic

Fix $\Delta = \{t_1[\kappa], \ldots, t_n[\kappa]\}^{\rightsquigarrow}$. We consider the first-order (relational) signature $\tau = \{\preceq, \mathcal{S}, \mathcal{O}, \mathcal{I}\}$ of type $(2, 2, 1, 1)$ and ω^+ as a τ structure by defining

- $\preceq^{\omega^+}$ as the natural order of ω^+,

Step 3. Translation into Logic

Fix $\Delta = \{t_1[\kappa], \ldots, t_n[\kappa]\}^{\rightsquigarrow}$. We consider the first-order (relational) signature $\tau = \{\preceq, \mathcal{S}, \mathcal{O}, \mathcal{I}\}$ of type $(2, 2, 1, 1)$ and ω^+ as a τ structure by defining

- $\preceq^{\omega^+}$ as the natural order of ω^+,
- $\mathcal{S}^{\omega^+} := \{\langle n, n+1\rangle \mid n \in \omega\} \cup \{\langle \omega, \omega\rangle\}$,

Step 3. Translation into Logic

Fix $\Delta = \{t_1[\kappa], \ldots, t_n[\kappa]\}^{\rightsquigarrow}$. We consider the first-order (relational) signature $\tau = \{\preceq, \mathcal{S}, \mathcal{O}, \mathcal{I}\}$ of type $(2, 2, 1, 1)$ and ω^+ as a τ structure by defining

- $\preceq^{\omega^+}$ as the natural order of ω^+,
- $\mathcal{S}^{\omega^+} := \{\langle n, n+1\rangle \mid n \in \omega\} \cup \{\langle \omega, \omega\rangle\}$,
- $\mathcal{O}^{\omega^+} := \{\omega\}$, and $\mathcal{I}^{\omega^+} := \{0\}$.

Step 3. Translation into Logic

Fix $\Delta = \{t_1[\kappa], \ldots, t_n[\kappa]\}^{\rightsquigarrow}$. We consider the first-order (relational) signature $\tau = \{\preceq, \mathcal{S}, \mathcal{O}, \mathcal{I}\}$ of type $(2,2,1,1)$ and ω^+ as a τ structure by defining

- $\preceq^{\omega^+}$ as the natural order of ω^+,
- $\mathcal{S}^{\omega^+} := \{\langle n, n+1\rangle \mid n \in \omega\} \cup \{\langle \omega, \omega\rangle\}$,
- $\mathcal{O}^{\omega^+} := \{\omega\}$, and $\mathcal{I}^{\omega^+} := \{0\}$.

If we consider the elements of Δ as variables, a Δ-prediagram $\delta \colon \Delta \to \omega^+$ is just a τ-valuation into ω^+. Moreover, we can translate the diagram conditions into a finite set Γ_Δ of quantifier-free first-order formulas.

Step 3. Translation into Logic

Fix $\Delta = \{t_1[\kappa], \dots, t_n[\kappa]\}^{\rightsquigarrow}$. We consider the first-order (relational) signature $\tau = \{\preceq, \mathcal{S}, \mathcal{O}, \mathcal{I}\}$ of type $(2, 2, 1, 1)$ and ω^+ as a τ structure by defining

- $\preceq^{\omega^+}$ as the natural order of ω^+,
- $\mathcal{S}^{\omega^+} := \{\langle n, n+1 \rangle \mid n \in \omega\} \cup \{\langle \omega, \omega \rangle\}$,
- $\mathcal{O}^{\omega^+} := \{\omega\}$, and $\mathcal{I}^{\omega^+} := \{0\}$.

If we consider the elements of Δ as variables, a Δ-prediagram $\delta \colon \Delta \to \omega^+$ is just a τ-valuation into ω^+. Moreover, we can translate the diagram conditions into a finite set Γ_Δ of quantifier-free first-order formulas.

For example, the condition

$$\forall t[\alpha], t[\beta] \in \Delta, \delta(\alpha) \leq \delta(\beta) \Rightarrow \delta(t[\alpha]) \leq \delta(t[\beta]) \tag{1}$$

yields the set $\{\alpha \preceq \beta \Rightarrow t[\alpha] \preceq t[\beta] \mid t[\alpha], t[\beta] \in \Delta\}$.

Step 3. Decidability via Logic

If we set fail $:= \{t_i[\kappa] \prec \kappa \mid 1 \leq i \leq n\}$ and take the conjunction over $\Gamma_\Delta \cup$ fail, we get a quantifier-free formula ψ_Δ.

Step 3. Decidability via Logic

If we set fail $:= \{t_i[\kappa] \prec \kappa \mid 1 \leq i \leq n\}$ and take the conjunction over $\Gamma_\Delta \cup$ fail, we get a quantifier-free formula ψ_Δ.

Proposition

Let $\delta\colon \Delta \to \omega^+$ be a Δ-prediagram. Then $\omega^+, \delta \models \psi_\Delta$ if, and only if, δ is a Δ-diagram such that $\delta(t_i[\kappa]) < \delta(\kappa)$ for each $i \in \{1, \ldots, n\}$.

Step 3. Decidability via Logic

If we set fail $:= \{t_i[\kappa] \prec \kappa \mid 1 \leq i \leq n\}$ and take the conjunction over $\Gamma_\Delta \cup$ fail, we get a quantifier-free formula ψ_Δ.

Proposition

Let $\delta\colon \Delta \to \omega^+$ be a Δ-prediagram. Then $\omega^+, \delta \models \psi_\Delta$ if, and only if, δ is a Δ-diagram such that $\delta(t_i[\kappa]) < \delta(\kappa)$ for each $i \in \{1, \ldots, n\}$.

Together with the Diagram Theorem we get that ψ_Δ is satisfiable in ω^+ if and only if $\mathbf{W} \not\models id \leq t_1 \vee \ldots \vee t_n$. From this the decidability follows, by a classical decidability result about ordinals (Läuchli and Leonard 1966).

Step 3. Translation into a Problem over $\mathbb{N}$

A structure which is more commonly available in satisfiability solvers (for example in the Z3 theorem prover) is the structure $\langle \mathbb{N}, \leq^{\mathbb{N}}, 0^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}} \rangle$, where $\leq^{\mathbb{N}}$ is the natural order, $0^{\mathbb{N}} = 0$, and $\mathcal{S}^{\mathbb{N}}$ is the successor relation.

Step 3. Translation into a Problem over $\mathbb{N}$

A structure which is more commonly available in satisfiability solvers (for example in the Z3 theorem prover) is the structure $\langle \mathbb{N}, \leq^{\mathbb{N}}, 0^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}} \rangle$, where $\leq^{\mathbb{N}}$ is the natural order, $0^{\mathbb{N}} = 0$, and $\mathcal{S}^{\mathbb{N}}$ is the successor relation.

We can explicitly translate the τ-formula ψ_Δ into a quantifier-free $\{\leq, 0, \mathcal{S}\}$-formula ϕ_Δ such that ψ_Δ is satisfiable in ω^+ if and only if ϕ_Δ is satisfiable in $\mathbb{N}$.

Step 3. Translation into a Problem over $\mathbb{N}$

A structure which is more commonly available in satisfiability solvers (for example in the Z3 theorem prover) is the structure $\langle \mathbb{N}, \leq^{\mathbb{N}}, 0^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}} \rangle$, where $\leq^{\mathbb{N}}$ is the natural order, $0^{\mathbb{N}} = 0$, and $\mathcal{S}^{\mathbb{N}}$ is the successor relation.

We can explicitly translate the τ-formula ψ_Δ into a quantifier-free $\{\leq, 0, \mathcal{S}\}$-formula ϕ_Δ such that ψ_Δ is satisfiable in ω^+ if and only if ϕ_Δ is satisfiable in $\mathbb{N}$.

Theorem

The time warp equation $id \leq t_1 \vee \cdots \vee t_n$ *is valid in* $\mathbf{W}$ *if, and only if, the quantifier-free formula* ϕ_Δ *is unsatisfiable in* $\mathbb{N}$*. Moreover, any valuation* $w \colon \Delta \to \mathbb{N}$ *such that* $\mathbb{N}, w \models \phi_\Delta$ *effectively yields a valuation* θ *of the time warp variables occurring in* $t_1 \vee \cdots \vee t_n$ *such that* $\mathbf{W}, \theta \models id \nleq t_1 \vee \cdots \vee t_n$*.*

Conclusion and Further Directions

We found a procedure to decide whether an equation holds in $\mathbf{W}$. From the proof an upper bound for the complexity of the decidability problem can be calculated, but the precise complexity is unknown.

Conclusion and Further Directions

We found a procedure to decide whether an equation holds in $\mathbf{W}$. From the proof an upper bound for the complexity of the decidability problem can be calculated, but the precise complexity is unknown.

Some further directions are

- Implementing the procedure.

Conclusion and Further Directions

We found a procedure to decide whether an equation holds in $\mathbf{W}$. From the proof an upper bound for the complexity of the decidability problem can be calculated, but the precise complexity is unknown.

Some further directions are

- Implementing the procedure.
- Finding a more suitable way to encode the decidability procedure.

Conclusion and Further Directions

We found a procedure to decide whether an equation holds in $\mathbf{W}$. From the proof an upper bound for the complexity of the decidability problem can be calculated, but the precise complexity is unknown.

Some further directions are

- Implementing the procedure.
- Finding a more suitable way to encode the decidability procedure.
- Finding a good equational basis for the equational theory to establish which variety of residuated lattices is generated by $\mathbf{W}$.

Conclusion and Further Directions

We found a procedure to decide whether an equation holds in $\mathbf{W}$. From the proof an upper bound for the complexity of the decidability problem can be calculated, but the precise complexity is unknown.

Some further directions are

- Implementing the procedure.
- Finding a more suitable way to encode the decidability procedure.
- Finding a good equational basis for the equational theory to establish which variety of residuated lattices is generated by $\mathbf{W}$.
- Generalizing the decidability proof to other residuated lattices of sup-preserving endomaps.

Thank you!

References

Adrien Guatto. A Generalized Modality for Recursion. In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '18. Oxford, United Kingdom: ACM, 2018, 482–491.

W.C. Holland and S.H. McCleary. *Solvability of the word problem in free lattice-ordered groups*. Houston J. Math. 5.1 (1979), 99–105.

H. Läuchli and J. Leonard. *On the elementary theory of linear order*. Fund. Math. 59 (1966), 109–116.

Luigi Santocanale. The Involutive Quantaloid of Completely Distributive Lattices. In: *RAMICS 2020*. Ed. by Uli Fahrenberg, Peter Jipsen, and Michael Winter. Vol. 12062. Cham: Springer, 2020, 286–301.