Abstract strategies and coherence

Relational and Algebraic Methods in Computer Science

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3^{rd} of November 2021

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Introduction

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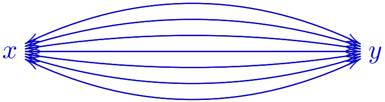

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- An ARS is **coherent** when any two parallel zig-zags are equivalent modulo some 'higher' relations.
  - Rewriting provides a constructive method for proving coherence,
  - and calculating a **generating set** of such relations.

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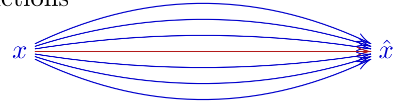

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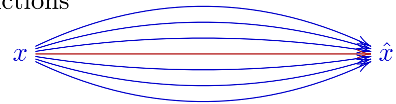

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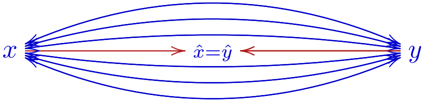

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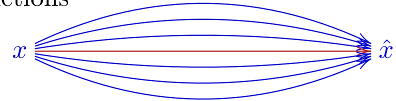

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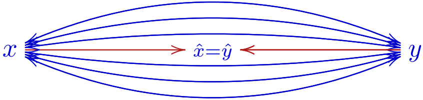
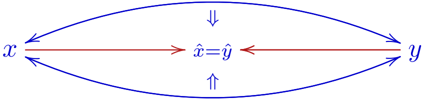


  - Proving coherence can be achieved by paving toward this canonical zig-zag:



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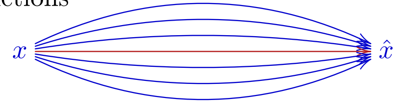

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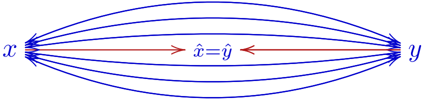

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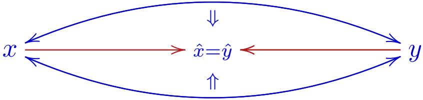

- This paving is generated by **local branchings**.

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- We introduce **globular 2-Kleene algebras**, extending MKA, as a natural setting for abstract coherence proofs.

Theorem (Abstract coherence theorem)

Let K be a Boolean globular 2-Kleene algebra (satisfying additional hypotheses) and $\phi \in K_1$ convergent. For any skeleton σ of $exh(\phi)$,

$$\overline{\phi} \odot_0 \phi \leq |A\rangle_1(\phi^{*_0} \odot_0 \overline{\phi}^{*_0}) \quad \Rightarrow \quad \phi^{\top_0} = (\phi + \overline{\phi})^{*_0} \leq |\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}).$$

Rewriting in modal Kleene algebras

Modal Kleene algebras

- A **dioid** is a tuple $K = (K,\ +,\ 0,\ \cdot\ ,\ 1)$ such that
  - $(K, +, 0)$ is a commutative monoid.
  - $(K, \cdot, 1)$ is a monoid.
  - Distributivity: $(x + y) \cdot z = x \cdot z + x \cdot y$ and $x \cdot (y + z) = x \cdot y + x \cdot z$.
  - Annhilator: $x \cdot 0 = 0 = 0 \cdot x$
  - Idempotence: $x + x = x$

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  - Idempotence: $x + x = x$
  - We equip K with a an **order** $\leq$ given by

$$x \leq y \qquad \Longleftrightarrow \qquad x + y = y$$

Modal Kleene algebras

- A **Kleene algebra** is a dioid K equipped with a map

$$(-)^* : \ K \to K$$

satisfying the following axioms for all $x, y, z \in K$:

$$1 + xx^* \leq x^*, \qquad y + xz \leq z \Rightarrow x^*y \leq z,$$
$$1 + x^*x \leq x^*, \qquad y + zx \leq z \Rightarrow yx^* \leq z.$$

Modal Kleene algebras

- A Kleene algebra K along with a map $ad : K \to K$ satisfying:

$$ad(x)x = 0, \quad ad(xy) \leq ad(x\, ad^2(y)), \quad ad^2(x) + ad(x) = 1,$$

for all $x, y \in K$ is called a **(Boolean) Kleene algebra with domain**. The map $d := ad^2$ is called the **domain map**.

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- In such a structure, the **domain algebra**

$$K_d := d(K) = \{\, x \in K | d(x) = x\}$$

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- We similarly axiomatise a notion of **codomain** $r : K \to K$.
- K is a **(Boolean) modal Kleene algebra (MKA)** when equipped with a domain and a codomain map.

$$K_d \; \overset{r(-)}{\longleftarrow} \!\!\!\! \underset{d(-)}{\overset{\longrightarrow}{\longleftarrow}} \; K$$

Modalities and conversion

Let K be a modal Kleene algebra and $x \in K$.

- We define the **forward diamond operator** given by x:

$$\begin{aligned} |x\rangle : K_d &\longrightarrow K_d \\ p &\longmapsto d(x \cdot p) \end{aligned}$$

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$$\overline{(x+y)} = \overline{x} + \overline{y}, \qquad \overline{(x \cdot y)} = \overline{y} \cdot \overline{x}, \quad \text{and} \quad \overline{(x^*)} = (\overline{x})^*.$$

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  Finally, conversion is **contracting** in the sense that

$$d(x) \leq x \cdot \overline{x}.$$

Rewriting properties in MKAs

Let K be a modal Kleene algebra and $x, y \in K$.

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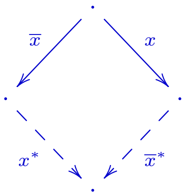

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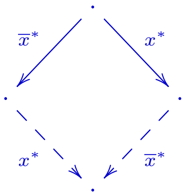

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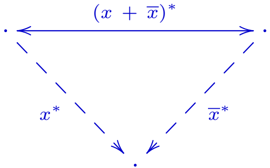

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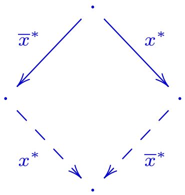

- We say that x is **convergent** when it both terminates and is confluent.

Formalisation of coherence

2-Kleene algebras

- A **globular 2-Kleene algebra** is a tuple

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- Multiplications satisfy the **weak exchange law**:

$$(A \odot_1 A') \odot_0 (B \odot_1 B') \leq (A \odot_0 B) \odot_1 (A' \odot_0 B').$$

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- (co-)Domains satisfy **absorption laws** $d_1 \circ d_0 = d_0$ and $r_1 \circ r_0 = r_0$. The domain algebra K_{d_i} will be denoted by K_i.

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- The 1-star is a **lax morphism** w.r.t. 0-multiplication by 1-dimensional elements, *i.e.* for all $A \in K$ and $\phi \in K_1$,

$$\phi \odot_0 A^{*_1} \leq (\phi \odot_0 A)^{*_1}, \quad \text{and} \quad A^{*_1} \odot_0 \phi \leq (A \odot_0 \phi)^{*_1}.$$

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- Finally, the (co-)domains satisfy **globularity conditions**:

$$d_0 \circ d_1 = d_0 \ \text{ and } \ d_0 \circ r_1 = d_0, \qquad d_1(A \odot_0 B) = d_1(A) \odot_0 d_1(B),$$
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$$K_0 \underset{d_0(-)}{\overset{r_0(-)}{\leftleftarrows\!\!\!\rightarrow}} K_1 \underset{d_1(-)}{\overset{r_1(-)}{\leftleftarrows\!\!\!\rightarrow}} K_2$$

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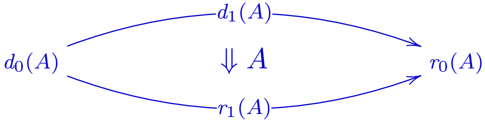
- An element $A \in K$ is represented graphically by:



Paving via modalities

- Just as in the case of MKAs, we obtain modalities for each multiplication:

$$\begin{aligned} |A\rangle_0 : K_0 &\longrightarrow K_0 \\ p &\longmapsto d_0(A \odot_0 p) \end{aligned} \qquad \begin{aligned} |A\rangle_1 : K_1 &\longrightarrow K_1 \\ \phi &\longmapsto d_1(A \odot_1 \phi) \end{aligned}$$

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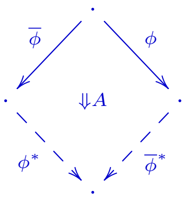

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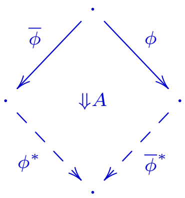

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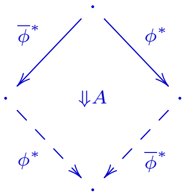

Paving via modalities

Let K be a globular 2-Kleene algebra, $A \in K$ and $\phi \in K_1$.

- We say that A is...
  - a **Church-Rosser filler** for ϕ if

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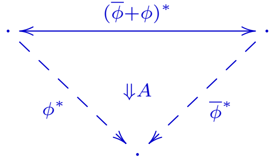

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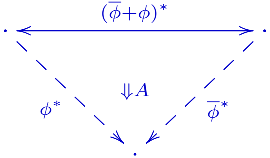

- When A is a filler for ϕ, the **total whiskering** of A is the element

$$\hat{A} := (\overline{\phi} + \phi)^{*_0} \odot_0 A \odot_0 (\overline{\phi} + \phi)^{*_0}$$

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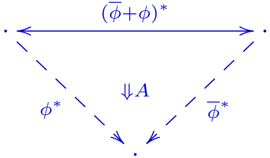

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- The **completion** of A is the element $\hat{A}^{*_1}$.

Strategies and coherence

Strategies

Let K be an MKA, $x \in K$ and $p \in K_d$.

- The **equivalence** generated by x is the element $x^\top := (x + \overline{x})^*$.
- The x**-saturation** of p is the element $|x^\top\rangle(p) \in K_d$.
- A **covering set** for x is an element $q \in K_d$ such that $|x^\top\rangle(q) \geq 1$.
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- Given a section s_0 of x, a **strategy for x relative to s_0** is:
  - a skeleton σ of $x^\top s_0$,
  - such that $s_0 \sigma \leq s_0$.

Convergence yields a strategy

Let K be a MKA and $x \in K$.

- The **exhaustion** of x is defined by:

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Proposition

Let K be a modal Kleene algebra and $x \in K$. If x is convergent, then

- *nf_x is a section of x.*
- *A skeleton σ of $exh(x)$ is a strategy for x with respect to nf_x.*

Abstract coherence theorem

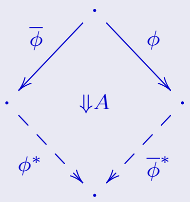
Theorem (Coherent normalising Newman's lemma)

Let K *be a globular* 2*-Kleene algebra such that*

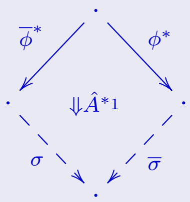
- $(K_0, +, 0, \odot_0, \mathbb{1}_0, \neg)$ *is a complete Boolean algebra,*
- K_1 *is continuous with respect to* 0*-restriction.*

Let $\phi \in K_1$ *be convergent and* σ *be a skeleton of* $exh(\phi)$*. We have*

$$\overline{\phi} \odot_0 \phi \leq |A\rangle_1(\phi^{*0} \odot_0 \overline{\phi}^{*0}) \quad \Rightarrow \quad \overline{\phi}^{*0} \odot_0 \phi^{*0} \leq |\hat{A}^{*1}\rangle_1(\sigma \odot_0 \overline{\sigma})$$



$\Rightarrow$



A *is a local confluence filler for* ϕ

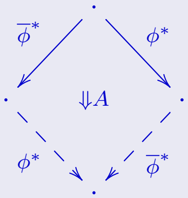
$\hat{A}^{*1}$ *paves branchings in* ϕ *to confluences in* σ

Abstract coherence theorem

Theorem (Abstract coherence theorem)

Let K be a Boolean globular 2-Kleene algebra satisfying the previous hypotheses and $\phi \in K_1$ convergent and a skeleton σ of $exh(\phi)$. We have

$$\overline{\phi}^{*0} \odot_0 \phi^{*0} \leq |A\rangle_1(\phi^{*0} \odot_0 \overline{\phi}^{*0}) \quad \Rightarrow \quad \phi^{\top_0} = (\phi + \overline{\phi})^{*0} \leq |\hat{A}^{*1}\rangle_1(\sigma \odot_0 \overline{\sigma}).$$



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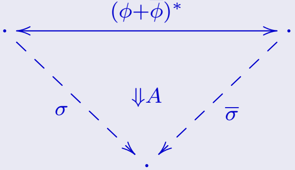

$\hat{A}^{*1}$ paves zig-zags in ϕ to confluences in σ.

Conclusion

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  - Define higher dimensional strategies,
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  - obtain a formalisation of calculating **resolutions** via rewriting,
- The end goal for ARS is a formalisation of **cofibrant replacement** via convergent rewriting.
- After finishing the case of ARS, move on to **string and term** rewriting systems:
  - Capture **contexts** via residuation,
  - thereby state and prove the **critical branching lemma**.

Thank you