A Variety Theorem for Relational Universal Algebra

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Chad Nester

(Tallinn University of Technology)

The Plan:

- Universal Algebra Refresher
- String Diagrams as Relational Syntax
- Relational Universal Algebra
- Variety Theorem & Morita Equivalence

Recall the theory of Commutative Monoids:

$$\Sigma = \{ m/2, e/0 \}$$

Subject to equations:

$$m(m(x,y),z) = m(x, m(y,z))$$
$$m(e(), x) = x$$
$$m(x,y) = m(y,x)$$

A <u>model</u> is a set X along with functions

$$m_X : X \times X \longrightarrow X \qquad e_X : 1 \longrightarrow X$$

which satisfy the equations.

A model morphism $X \xrightarrow{f} Y$ is a function satisfying:

$$f(m_X(x,y)) = m_Y(f(x), f(y))$$

$$f(e_X()) = e_Y()$$

Resulting in the category of monoids and monoid homomorphisms.

More Abstractly:

- Theories $\leftrightsquigarrow$ Categories $\mathbb{X}$ w/ finite products

- Models $\leftrightsquigarrow$ Product-Preserving functors
$$\mathbb{X} \longrightarrow \underline{Set}$$

- Model Morphisms $\leftrightsquigarrow$ Natural transformations

Categories that arise as the models and model morphisms of some theory are called <u>varieties</u>.

String Diagrams / Symmetric Monoidal Categories:

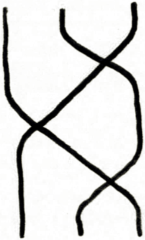
$f;g \leftrightsquigarrow$ 

$f\otimes g \leftrightsquigarrow$ 

$\sigma \leftrightsquigarrow$ 

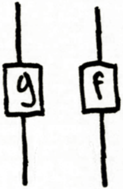
$1 \leftrightsquigarrow$ 

"Same" Diagram $\Leftrightarrow$ Same Arrow:

 = 

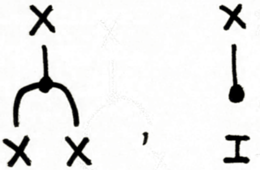
$(\sigma\otimes 1)(1\otimes\sigma)(\sigma\otimes 1)$
$= (1\otimes\sigma)(\sigma\otimes 1)(1\otimes\sigma)$

and

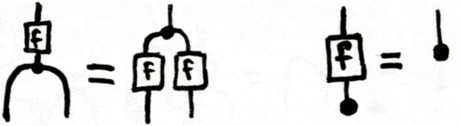
= 

$(1\otimes f)\sigma(1\otimes g)\sigma = g\otimes f$

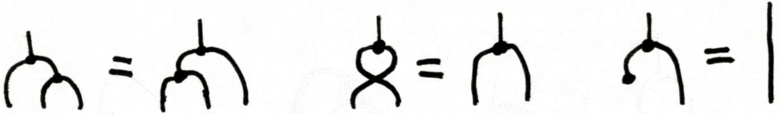
Finite Products ⟺ SMC w/ natural Commutative Comonoid structure:



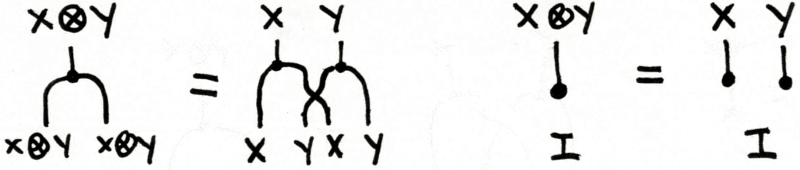
satisfying:



(naturality)

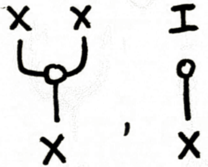


(Commutative Comonoid)

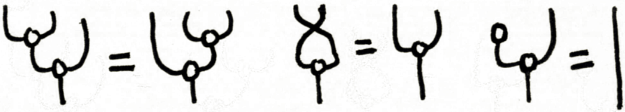


(Coherence)

Theory of Commutative Monoids may be given as:



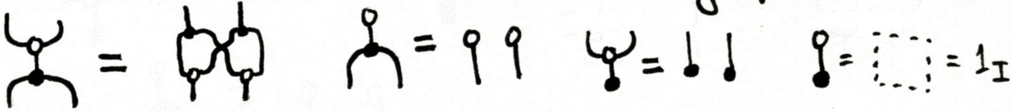
:



(monoidal signature)

(equations over free Cat. w̄ finite Products on the signature)

Notice that this includes the following equations implicitly:



The Associated Variety is again monoids & monoid homomorphisms

Now String diagrams Correspond to terms:

$e \leftrightsquigarrow$ 

$x_1 \circ x_2 \leftrightsquigarrow$ 

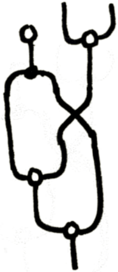
$(x_1 \circ x_2) \circ x_1 \leftrightsquigarrow$ 

Composition is Substitution:

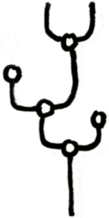
$$(x_1 \circ x_2) \circ x_1 \begin{bmatrix} x_1 \mapsto e \\ x_2 \mapsto x_1 \circ x_2 \end{bmatrix}$$

$$=$$

$$(e \circ (x_1 \circ x_2)) \circ e$$



$=$



Theorem: (Adámek, Lawvere, Rosický)

Two algebraic theories present the same variety iff splitting idempotents in each yields equivalent categories.

"Morita Equivalence"

(A consequence of their characterisation of varieties)

What about theories whose operations are <u>relations</u>?

That is, models in <u>Rel</u> instead of <u>Set</u>.

- Classical syntax $\leftrightsquigarrow$ Finite products.

<u>Rel</u> does not have finite products in the right way ($\uplus$, not $\times$).

... So classical term syntax won't really work!

String diagram syntax for relations:

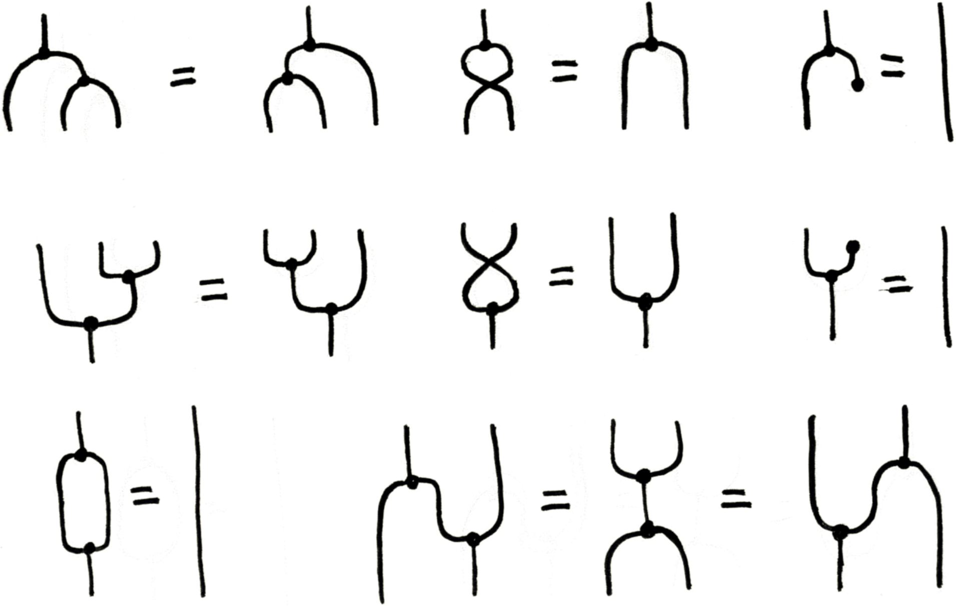
$\{(a,(a,a)) \mid a \in X\}$

$\{(a,*) \mid a \in X\}$

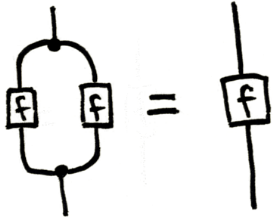
$\{((a,a),a) \mid a \in X\}$

$\{(*,a) \mid a \in X\}$

Forming a Commutative Special Frobenius Algebra:

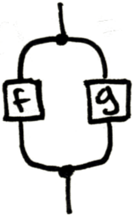


Further, we ask that for all f:



which determines a meet & poset-enrichment:

$f \cap g =$



$f \leq g \Leftrightarrow f \cap g = f$

g <u>includes</u> f

A **Relational Algebraic Theory** is a symmetric monoidal category with this structure.

A **model** is a structure-preserving functor

$$\mathbb{X} \xrightarrow{F} \underline{\mathrm{Rel}}$$

Aside:

Categories with this structure were introduced by Carboni & Walters in 1987, under the name "Cartesian bicategories of Relations".

A **Model Morphism** $\alpha : F \to G$ is a lax transformation.

That is, a family of arrows $\alpha_X : FX \to GX$

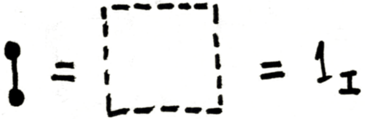
Satisfying:

$$X \xrightarrow{f} Y \quad \Rightarrow \quad \begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\leq"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

where $\leq$ is the inclusion ordering:

$$f \leq g \Leftrightarrow f \cap g = f \quad \text{(Introduced earlier)}$$

For example the theory of <u>nonempty sets</u> is the free such category on one generating object and no generating arrows subject to:



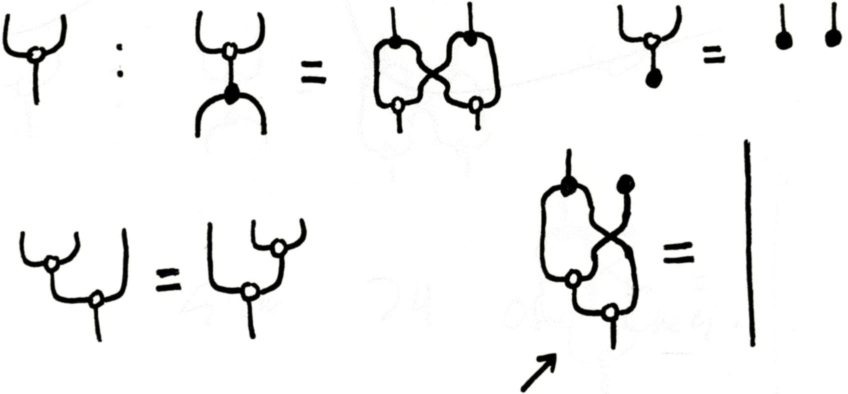
Models are nonempty sets:

 $= \{(*,a) \mid a \in X\} ; \{(a,*) \mid a \in X\}$

$= \{(*,*) \mid \exists a \in X\}$

Model morphisms are functions.

The theory of <u>regular semigroups</u> is generated by:

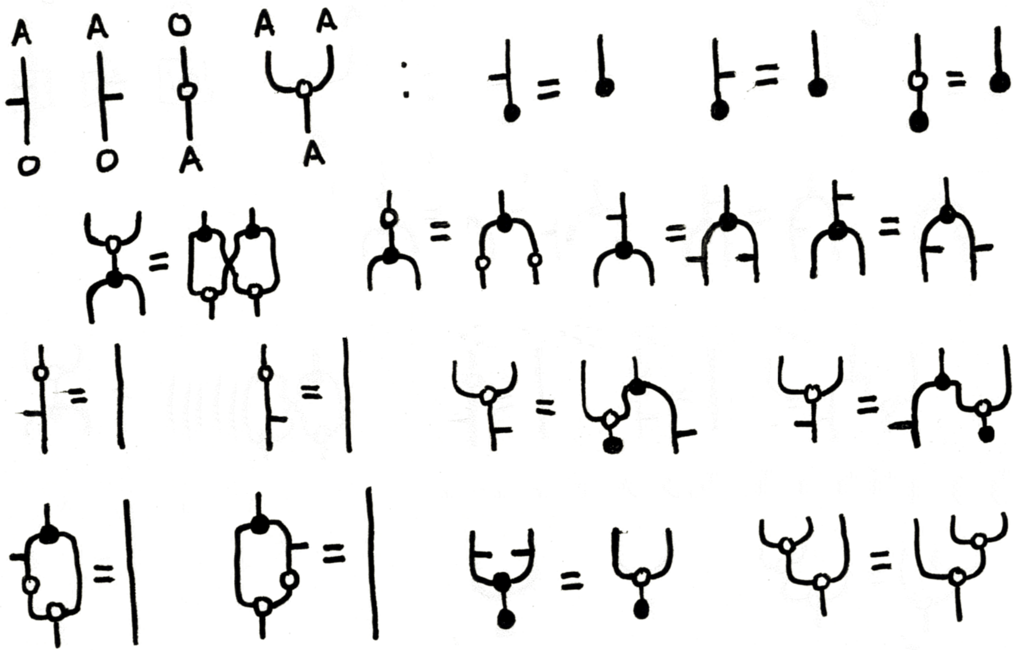


$\forall a \in S. \exists b \in S.\ aba = a$ – "Regularity"

Models are regular semigroups.

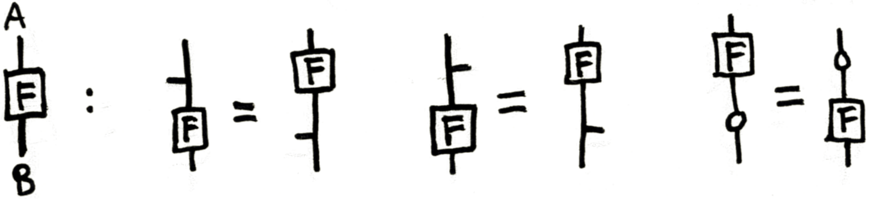
Model morphisms are semigroup homomorphisms.

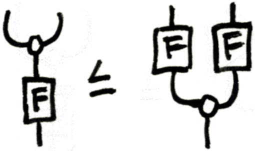
<u>Categories</u> are captured by the 2-sorted theory:



Models are (small) categories.

Model morphisms are functors. Here it is important that we use lax transformations:





If f, g are composable then so are Ff, Fg and $F(fg) = Ff\,Fg$.

Variety Theorem

Theorem:

The categories that arise as the models and model morphisms of some relational algebraic theory are precisely the "definable categories".

Kuber & Rosický 2016
Lack & Tendas 2019

Morita-Equivalence

<u>Theorem:</u>

Two relational algebraic theories present the same definable category if and only if splitting partial equivalence relations yields equivalent categories.

Possibly just splitting idempotents (Categories, Allegories 2.16(11))

Formal Machinery

$$\mathrm{RAT} \xrightarrow{\mathrm{Split}_{\mathrm{cor}}} \mathrm{RAT}_{\mathrm{tab}} \xrightarrow[\sim]{\mathrm{Map}} \mathrm{REG} \xrightarrow{\mathrm{REG}(-,\,\mathrm{Set})} \mathrm{DEF}^{\mathrm{op}}$$

Split coreflexives

Subcategory of maps

Regular presheaves

Formal Machinery

$$
\begin{tikzcd}
\mathrm{RAT} \arrow[r, "\mathrm{Split}_{cor}"] \arrow[dr, "\mathrm{Split}_{per}"'] & \mathrm{RAT}_{tab} \arrow[rr, "\mathrm{Map}", "\sim"'] \arrow[d, "\mathrm{Split}_{eq}"] & & \mathrm{REG} \arrow[r, "\mathrm{REG}(-,\mathrm{Set})"] \arrow[d, "ex/reg"'] & \mathrm{DEF}^{op} \\
& \mathrm{RAT}_{eff} \arrow[rr, "\sim", "\mathrm{Map}"'] & & \mathrm{EX} \arrow[ur, "\sim", "\mathrm{REG}(-,\mathrm{Set})"'] &
\end{tikzcd}
$$

References:

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