A Variety Theorem for
Relational Universal Algebra

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Chad Nester
(Tallinn University of Technology)
The Plan:

- Universal Algebra Refresher
- String Diagrams as Relational Syntax
- Relational Universal Algebra
- Variety Theorem & Morita Equivalence
Recall the theory of commutative monoids:

\[ \Sigma = \{ m/a, e/o \} \]

Subject to equations:

\[ m(m(x,y),z) = m(x,m(y,z)) \]
\[ m(e(),x) = x \]
\[ m(x,y) = m(y,x) \]
\[ m(x,x) = m(y,y) \]
A model is a set $X$ along with functions

$$m_x: X \times X \to X \quad e_x: 1 \to X$$

which satisfy the equations.

A model morphism $X \xrightarrow{f} Y$ is a function satisfying:

$$f(m_x(x, y)) = m_y(f(x), f(y))$$

$$f(e_x()) = e_y()$$

Resulting in the category of monoids and monoid homomorphisms.
More Abstractly:

- Theories $\leftrightarrow$ Categories with finite products
- Models $\leftrightarrow$ Product-Preserving functors
  
  $\mathbb{X} \rightarrow \text{Set}$

- Model Morphisms $\leftrightarrow$ Natural transformations

Categories that arise as the models and model morphisms of some theory are called varieties.
String Diagrams / Symmetric monoidal Categories:

\[ f \circ g \leftrightarrow \begin{array}{c} f \\ g \end{array} \quad f \circ g \leftrightarrow f \otimes g \quad \varepsilon \leftrightarrow X \quad 1 \leftrightarrow \]

"Same" Diagram \[ \leftrightarrow \] Same Arrow:

\[ \begin{array}{ccc} = & \begin{array}{c} f \\ g \end{array} \\ & \end{array} \quad \text{and} \quad \begin{array}{ccc} = & g \\ & \quad f \end{array} \]

\[(\varepsilon \otimes 1)(1 \otimes \varepsilon)(\varepsilon \otimes 1) = (1 \otimes \varepsilon)(\varepsilon \otimes 1)(1 \otimes \varepsilon)\]

\[(1 \otimes f) \circ (1 \otimes g) = g \otimes f\]
Finite Products $\iff$ SMC $\iff$ Natural Commutative Comonoid Structure:

Satisfying:

- (naturality)

- (Commutative Comonoid)

- (Coherence)
Theory of Commutative Monoids may be given as:

\[
\begin{align*}
\times & \quad \times & \quad I \\
\bullet & \quad \circ & \quad \circ
\end{align*}
\]

(monomial signature)

(equations over free cat. to finite products on the signature)

Notice that this includes the following equations implicitly:

\[
\begin{align*}
\bullet & \quad \circ \\
\circ & \quad \bullet
\end{align*}
\]

The associated variety is again monoids & monoid homomorphisms
Now string diagrams correspond to terms:

\[ e \leftrightarrow \circ \quad x_1 \circ x_2 \leftrightarrow \bigcirc \quad (x_1 \circ x_2) \circ x_1 \leftrightarrow \text{Diagram} \]

Composition is substitution:

\[
(x_1 \circ x_2) \circ x_1 \left[ \begin{array}{c}
\xrightarrow{x_1 \mapsto e} \\
\xrightarrow{x_2 \mapsto x_1 \circ x_2}
\end{array} \right] = (e \circ (x_1 \circ x_2)) \circ e
\]
Theorem: (Adámek, Lawvere, Rosicky)

Two algebraic theories present the same variety iff splitting idempotents in each yields equivalent categories.

"Morita Equivalence"

(A consequence of their characterization of varieties)
What about theories whose operations are relations?

That is, models in $\text{Rel}$ instead of $\text{Set}$.

- Classical syntax $\leftrightarrow$ Finite products.

$\text{Rel}$ does not have finite products in the right way ($\prod$, not $\times$).

... So classical term syntax won't really work!
String diagram syntax for relations:

\[ 3(a, (a, a)) \mid a \in \mathbb{X}^3 \]

\[ 3(a, *) \mid a \in \mathbb{X}^3 \]

\[ 3((a,a), a) \mid a \in \mathbb{X}^3 \]

\[ 3(*, a) \mid a \in \mathbb{X}^3 \]

\[ I = \exists \times \exists \]

\[ \exists \times \exists \]

\[ \exists \times \exists \]
Forming a Commutative Special Frobenius Algebra:
Further, we ask that for all \( f \):

\[
\begin{array}{c}
\text{ } \\
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

which determines a meet & poset-enrichment:

\[
f \leq g \iff f \circ g = f
\]

\[
g \text{ includes } f
\]
A Relational Algebra Theory is a symmetric monoidal category with this structure.

A model is a structure-preserving functor

\[ X \xrightarrow{F} \text{Rel} \]

Aside:

Categories with this structure were introduced by Carboni & Walters in 1987, under the name "Cartesian bicategories of Relations".
A morphism \( \alpha : F \to G \) is a lax transformation. That is, a family of arrows \( \alpha_x : Fx \to Gx \) satisfying:

\[
\begin{array}{c}
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
F & \downarrow & \downarrow \alpha_f \\
FY & \xrightarrow{\alpha_Y} & GY
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
FX & \xrightarrow{\alpha_x} & GX \\
Ff & \downarrow \leq & Gf \\
FY & \xrightarrow{\alpha_Y} & GY
\end{array}
\end{array}
\]

where \( \leq \) is the inclusion ordering:

\[
f \leq g \iff Ff \circ g = F (\text{Introduces earlier})
\]
For example, the theory of nonempty sets is the free such category on one generating object and no generating arrows subject to:

\[ \mathbb{I} = \begin{array}{c} \bullet \\ \hline \hline \hline \hline \end{array} = \mathbf{1}_\mathbb{I} \]

Models are nonempty sets:

\[ \mathbb{I} = \{ (\ast, a) \mid a \in X \} \ni (a, \ast) \mid a \in X \]

\[ = \{ (\ast, \ast) \mid \exists a \in X \} \ni \]

Model morphisms are functions.
The theory of **regular semigroups** is generated by:

\[
\begin{align*}
\varepsilon & : \quad \varepsilon = \varepsilon \\
\gamma & = \varepsilon \\
\delta & = \varepsilon
\end{align*}
\]

\[\varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon = \text{"Regularity"}\]

Models are regular semigroups.

Model morphisms are semigroup homomorphisms.
Categories are captured by the 2-sorted theory:
Models are (Small) Categories.

Model morphisms are functors. Here it is important that we use lax transformations:

\[
\begin{align*}
F : & F = F \\
A & \quad B
\end{align*}
\]

\[
\begin{align*}
F & = F \\
F & = F \\
\end{align*}
\]

\[
\begin{align*}
\phi & = \phi \\
\end{align*}
\]

If \( F, g \) are composable then so are \( Ff, Fg \) and \( F(fg) = FfFg \).
Variety Theorem

Theorem:
The categories that arise as the models and model morphisms of some relational algebraic theory are precisely the "definable categories".

Kuben & Rosticky 2016
Lack & Tendas 2019
Morita-Equivalence

Theorem:
TWO relational algebraic theories present the same definable category if and only if splitting partial equivalence relations yields equivalent categories.

Possibly just splitting idempotents (Categories, Allegories 3.16 (11))
Formal Machinery

\[ \text{RAT} \xrightarrow{\text{Split cor}} \text{RAT} \xrightarrow{\text{Map}_{\mathbb{N}}} \text{REG} \xrightarrow{\text{REG}(-, \mathbb{Z}^+)} \text{DEF} \]

- Split cor
- Coreflexives
- Map
- Subcategory of maps
- Regular presheaves
Formal Machinery

RAT \xrightarrow{\text{Split cmp}} RAT \xrightarrow{\text{tab}} \sim \xrightarrow{\text{Map}} \xrightarrow{\text{ex/reg}} \xrightarrow{\sim} \xrightarrow{\text{REG}} \xrightarrow{\sim} \xrightarrow{\text{DEFOP}} 

\xrightarrow{\text{ex/reg}} \xrightarrow{\text{REG}} \xrightarrow{\sim} \xrightarrow{\text{REG(-/SCA)}} 

\xrightarrow{\text{Map}} 

\xrightarrow{\text{Map}} 

\xrightarrow{\text{Map}}

\xrightarrow{\text{Map}}
References:
