Girard semigroups in autonomous category

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Motivations

Theorem (Eklund *et al* 2018, Santocanale 2020)

Let L be a complete lattice. The following are equivalent:

- L is a completely distributive lattice.
- The set of endomorphisms of L is a Girard quantale.

Theorem (Raney 1960, Higgs and Rowe 1989)

The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattice.

Motivations

Conjecture

Let L be an object of an autonomous category (symmetric monoidal closed). The following are equivalent:

- L is nuclear.
- The object of endomorphisms of L is a Girard semigroup.

Theorem (Raney 1960, Higgs and Rowe 1989)

The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattices.

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Quantales

Definition

A *quantale* $(Q, \star)$ is a complete lattice Q with a law

$$\star : Q \times Q \to Q$$

associative which distributes over the sup:

$$(\bigvee_{i \in I} x_i) \star y = \bigvee_{i \in I} (x_i \star y) \quad \text{and} \quad x \star (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \star y_i).$$

Remark

A quantale is a semigroup in the category **Slatt**.

Important examples

- If $(S, \star)$ is a semigroup, $(P(S), \star)$ is the free quantale over S.
- The set of endomorphisms $([L, L], \circ)$ over L in **Slatt**.

Right and left implications

The maps $(x \star -) : Q \to Q$ and $(- \star y) : Q \to Q$ are sup-preserving. They have both have a right adjoint:

$$x \multimap - \quad \text{and} \quad - \circ\!\!- \, y.$$

Which gives the following equivalences:

$$\frac{\dfrac{x \star y \leq z}{y \leq x \multimap z}}{x \leq z \circ\!\!- \, y}$$

Girard quantales

Usual definition

A *Girard quantale* is a tuple $(Q, \star, 0)$ with 0 a cyclic and dualizing element. That is, we have, for all x in Q,

$$\begin{aligned} x \multimap 0 = 0 \multimapinv x & \quad \text{(cyclic)} \\ x = (x \multimap 0) \multimap 0 & \quad \text{(dualizing in the cyclic case).} \end{aligned}$$

We write $(-)^{\perp} = (- \multimap 0) : Q \to Q^{\mathrm{op}}$.

Remark

In a Girard quantale $(Q, \star, 0)$ we have for all x and y in Q,

$$x \multimap y = x^{\perp} \parr y \quad \text{and} \quad y \multimapinv x = y \parr x^{\perp}$$
$$\text{with} \quad x \parr y = (y^{\perp} \star x^{\perp})^{\perp}$$

Girard quantale, another definition

Definition

A *Girard quantale* is a tuple $(Q, \star, (-)^{\perp})$ with $(-)^{\perp} : Q \to Q^{\mathrm{op}}$ a monotone map such that for all x and y in Q we have

1. $x = (x^{\perp})^{\perp}$
2. $x \multimap y = x^{\perp} ⅋ y$ and $y \circ\!- x = y ⅋ x^{\perp}$

   with

$$x ⅋ y = (y^{\perp} \star x^{\perp})^{\perp}$$

Girard quantale, another definition

Proposition

Let $(Q, \star)$ be a quantale. The following are equivalent:

1. There exists a map $(-)^\perp : Q \to Q^{\mathrm{op}}$ turning $(Q, \star, (-)^\perp)$ into a Girard quantale.
2. There exists a map $\langle -, - \rangle : Q \times Q \to 2$ which is sup-preserving in both variables such that

$$\langle xy, z \rangle = \langle x, yz \rangle$$

and such that the map $(-)^\perp : Q \to Q^{\mathrm{op}}$ defined by

$$x^\perp = \bigvee \{ y \mid \langle x, y \rangle = \perp \}$$

is an involution.

The unit of a Girard quantale

Proposition

Let $(Q, \star, (-)^{\perp})$ be a Girard quantale.

- If 0 is a cyclic and dualizing element then $(Q, \star)$ is unital.
- If $(Q, \star)$ has a unit 1 then $1^{\perp}$ is cyclic and dualizing.

Remark

If $(Q, \star)$ is a unitless quantale, then the Chu construction over it is a unitless Girard quantale.

Tight maps

Definition

For a map $f : L \longrightarrow L$, we define the two Raney's transforms:

$$f^{\vee}(x) := \bigvee_{x \not\leq t} f(t) \quad \text{and} \quad f^{\wedge}(x) := \bigwedge_{t \not\leq x} f(t)\,.$$

We write $[L, L]^t = \{f : L \to L \mid f^{\wedge\vee} = f\}$ the set of *tight maps*.

Remark

$[L, L]^t$ is the image of $(-)^{\vee} : [L, L]_{\wedge} \longrightarrow [L, L]$.

Proposition (LS and CL)

For every complete lattice L, $([L, L]^t, \circ, (-)^{\perp})$ is a Girard quantale with $f^{\perp} = l(f^{\wedge})$.

The theorem we aim to generalize

Definition

A complete lattice L is *completely distributive* if for all $\{x_{ij}\}_{i\in I,\, j\in J}$,

$$\bigwedge_{i\in I}\Big(\bigvee_{j\in J} x_{ij}\Big) = \bigvee_{\alpha: I\to J}\Big(\bigwedge_{i\in I} x_{i\alpha(i)}\Big).$$

Theorem

Let L be a complete lattice. The following are equivalent:

1. The lattice L is completely distributive.
2. $[L, L]^t = [L, L]$ (Raney, 1960)
3. There is a unique sup-preserving map $0 : L \to L$ such that $([L, L], \circ, 0)$ is a Girard quantale. (Eklund *et al* 2018, Santocanale 2020)
4. *The Girard quantale* $([L, L]^t, \circ, (-)^\perp)$ *has a unit.* (LS and CL)

Symmetric monoidal closed categories

Definition

A symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is *closed* if there is a natural bijection:

$$\frac{X \otimes Y \longrightarrow Z}{Y \longrightarrow [X, Z]}$$

For the following, $\mathcal{C}$ will be an autonomous category.
We write $(-)^* = [-, I] : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$.

Examples

- Standard examples: **Set**, k-**Vect**, a commutative unital quantale, *etc.*
- Our main example: the category **Slatt**.

The category of semigroups over $\mathcal{C}$

Objects of $\mathbf{Sem}_{\mathcal{C}}$: pairs (S, m) such that

$$\begin{array}{ccc} S \otimes S \otimes S & \xrightarrow{S \otimes m} & S \otimes S \\ {\scriptstyle m \otimes S}\downarrow & & \downarrow{\scriptstyle m} \\ S \otimes S & \xrightarrow[m]{} & S. \end{array}$$

Morphisms of $\mathbf{Sem}_{\mathcal{C}}$: arrows $f : S_1 \longrightarrow S_2$ such that

$$\begin{array}{ccc} S_1 \otimes S_1 & \xrightarrow{f \otimes f} & S_2 \otimes S_2 \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m_2} \\ S_1 & \xrightarrow[f]{} & S_2. \end{array}$$

Girard semigroups

Definition

A *Girard semigroup* is a tuple $(S, m, \langle -, - \rangle)$ where

- (S, m) is a semigroup;
- $\langle -, - \rangle : S \otimes S \longrightarrow I$ is called a *pairing*.
  Let $(-)^{\perp} : S \longrightarrow S^{*}$ be the transpose of $\langle -, - \rangle$.
  We require for the pairing to be:
  1. associative w.r.t the multiplication *i.e*:

$$\begin{array}{ccc} S \otimes S \otimes S & \xrightarrow{S \otimes m} & S \otimes S \\ {\scriptstyle m \otimes S}\downarrow & & \downarrow{\scriptstyle \langle -,- \rangle} \\ S \otimes S & \xrightarrow[\langle -,- \rangle]{} & I; \end{array}$$

  2. *symmetric*: $(-)^{\perp} = (\langle -, - \rangle \circ \sigma_{S,S})^{\sharp}$;
  3. *strong*: $(-)^{\perp}$ is an isomorphism.

Nuclearity

Definition

For every object L of $\mathcal{C}$, there exists a canonical arrow

$$\mathrm{mix}_L : L^* \otimes L \longrightarrow [L, L].$$

An object L is *nuclear* if mix_L is an isomorphism.

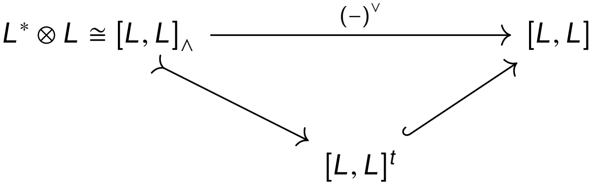
Theorem **(Raney 1960, Higgs and Rowe 1989)**

The nuclear objects of **Slatt** are exactly the completely distributive lattices.

Result

Theorem (LS and CL)

Let L be a complete lattice. The image of the Raney's transform $(-)^{\vee} : [L, L]_{\wedge} \to [L, L]$ can always be endowed with a Girard quantale structure.



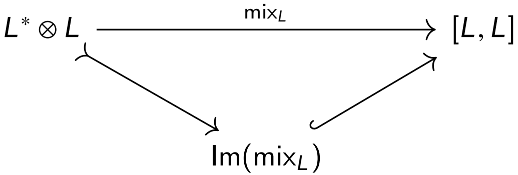
Corollary

If L is completely distributive then $[L, L]$ can always be endowed with a Girard quantale structure.

Result

Theorem (LS and CL)

Let $\mathcal{C}$ be an autonomous category such that $\mathbf{Sem}_{\mathcal{C}}$ has an epi-mono factorization system and L an object of $\mathcal{C}$.
The image of mix_L can always be endowed with a Girard semigroup structure.



Corollary

If L is nuclear then $[L, L]$ can always be endowed with a Girard semigroup structure.

Conclusion

Results

- The quantale of tight maps is always a (non necessary unital) Girard quantale.
- A definition of Girard semigroups in autonomous categories.
- One implication of the conjecture: if L is nuclear then $[L, L]$ is endowed with a structure of Girard semigroup.

What we will do next

- Find out whether if the converse is true.
- Understand what can be said about Frobenius semigroups *i.e* without the cyclicity.
- Connect with linear logic semantic.

Thank you for your attention !

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