Change of Base using Arrow Categories

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Motivation

We use arrow categories to formulate fuzzy concepts and constructions within the language of abstract categories of relations.

1. Arrow categories form an abstract theory of L-valued relation where L is a complete Heyting algebra.
2. We present an algebraic framework for handling a change of base for L-relations, i.e., a functor from $\mathcal{A}_1$ to $\mathcal{A}_2$.
3. Important to obtain a canonical embedding L-relations into type-2 L-relations, i.e., relations with membership values from $L \to L$, which is essential in the construction of type-2 fuzzy controllers.
   - The arrow category of type-2 L-relations is given by the Kleisli category of a (relational) monad based on the product functor.
   - The embedding maps an element u of L to the constant function $\bar{u} : L \to L$ defined by $\bar{u}(x) = u$.

Dedekind Categories

Definition

A Dedekind category $\mathcal{R}$ is a category satisfying the following:

1. For all objects A and B the collection $\mathcal{R}[A,B]$ is a complete distributive lattice with operations $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$.
2. There is a monotone operation $\breve{}$ (called conversion) so that for all relations $Q : A \to B$ and $R : B \to C$:
$$(Q;R)^{\smile} = R^{\smile};Q^{\smile}, \quad (Q^{\smile})^{\smile} = Q.$$
3. For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law holds:
$$Q;R \sqcap S \sqsubseteq Q;(R \sqcap Q^{\smile};S).$$
4. For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) so that for all $Q : A \to B$ the following holds:
$$Q;R \sqsubseteq S \iff Q \sqsubseteq S/R.$$

Special L-fuzzy Relations I

Relations corresponding to elements in L:

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$$\begin{pmatrix} a & a & a \\ a & a & a \end{pmatrix}$$

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scalar relations

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scalar relations

Definition

1. A relation $J : A \to B$ is called an ideal iff $\pi_{AA}; J; \pi_{BB} = J$.
2. A relation $\alpha : A \to A$ is called a scalar on A iff $\alpha \sqsubseteq \mathbb{I}_A$ and $\pi_{AA}; \alpha = \alpha; \pi_{AA}$.

Arrow Categories I

Definition

An arrow category $\mathcal{A}$ is a non-trivial Dedekind category ($\mathbb{T}_{AB} \neq \perp\!\!\!\perp_{AB}$ for all objects A and B) together with two operations $^{\uparrow}$ and $^{\downarrow}$ satisfying the following:

1. $R^{\uparrow}, R^{\downarrow} : A \to B$ for all $R : A \to B$.
2. $(^{\uparrow}, ^{\downarrow})$ is a Galois correspondence.
3. $(R^{\smile}; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile}; S^{\downarrow}$ for all $R : B \to A$ and $S : B \to C$.
4. $(Q \sqcap R^{\downarrow})^{\uparrow} = Q^{\uparrow} \sqcap R^{\downarrow}$ for all $Q, R : A \to B$.
5. If $\alpha_A \neq \perp\!\!\!\perp_{AA}$ is a non-zero scalar, then $\alpha_A^{\uparrow} = \mathbb{I}_A$.

Arrow Categories II

Theorem

Let $\mathcal{A}$ be an arrow category. Then we have:

1. *$\mathcal{A}$ is uniform, i.e., we have $\Pi_{AB};\Pi_{BC} = \Pi_{AC}$ for all objects A, B and C.*

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2. *The complete Heyting algebras $\mathrm{Sc}(A)$ and $\mathrm{Sc}(B)$ of scalar relations on any two two objects A and B are isomorphic. We denote this lattice by $\mathrm{Sc}[\mathcal{A}]$.*

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3. *The substructure $\mathcal{A}^{\downarrow}$ of crisp relations, i.e., all relations R with $R^{\downarrow} = R$ (or equivalently $R^{\uparrow} = R$), is a Dedekind category satisfying the so-called Tarski rule, i.e., the equivalence $R \neq \perp\!\!\!\perp_{AB}$ iff $\pi_{CA};R;\pi_{BD} = \pi_{CD}$.*

Important Constructions I

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For L-relations any singleton set is a unit.

In addition, for arrow categories $\pi_{11} = \mathbb{I}_1$ is sufficient since for every A we have

$$\begin{aligned} \pi_{A1};\pi_{1A} &= \pi_{AA} && \mathcal{A} \text{ uniform} \\ &\sqsupseteq \mathbb{I}_A. \end{aligned}$$

Important Constructions II

Definition

An object $A \times B$ together with relations $\pi : A \times B \to A$ and $\rho : A \times B \to B$ is called a relational product of A and B iff

$$\pi, \rho \text{ are crisp}, \quad \pi^{\smile};\pi \sqsubseteq \mathbb{I}_A, \quad \rho^{\smile};\rho \sqsubseteq \mathbb{I}_B, \quad \pi^{\smile};\rho = \mathbb{T}_{AB}, \quad \pi;\pi^{\smile} \sqcap \rho;\rho^{\smile} = \mathbb{I}_{A \times B}.$$

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$$\pi = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Important Constructions III

Definition

An object L^A together with a relation $\varepsilon : A \to L^A$ is called a fuzzy relational power iff

$$\mathrm{syQ}(\varepsilon, \varepsilon)^{\downarrow} \sqsubseteq \mathbb{I}_{L^A} \quad \text{and} \quad \mathrm{syQ}(R, \varepsilon)^{\downarrow} \text{ is total for every } R : A \to B.$$

Important Constructions III

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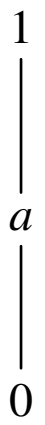
$L =$



$$\varepsilon = \begin{pmatrix} 0 & 0 & 0 & a & a & a & 1 & 1 & 1 \\ 0 & a & 1 & 0 & a & 1 & 0 & a & 1 \end{pmatrix}$$

Notation: $\Lambda(R) = \mathrm{syQ}(R^{\smile},\varepsilon)^{\downarrow} : A \rightarrow L^B$ if $R : A \rightarrow B$.

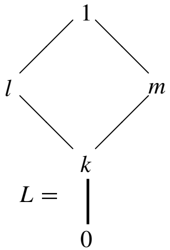
Internal Version of L

The object L^1 acts as an internal version of the lattice L.

Example:

$$\varepsilon = \begin{pmatrix} 0 & k & l & m & 1 \end{pmatrix}$$

$L =$



Every scalar $\alpha : 1 \to 1$ corresponds to a crisp map $p : 1 \to L$ (also called crisp point), i.e., $\Lambda(\alpha);\varepsilon^{\smile} = \alpha$ for every scalar $\alpha : 1 \to 1$ and $\Lambda(p;\varepsilon^{\smile}) = p$ for every crisp point $p : 1 \to L$.

The crisp order relation $(\varepsilon\backslash\varepsilon)^{\downarrow}$ corresponds to the order on scalars, i.e., $\alpha \sqsubseteq \beta$ iff $\mathrm{syQ}(\beta,\varepsilon)^{\downarrow} \sqsubseteq \mathrm{syQ}(\alpha,\varepsilon)^{\downarrow};(\varepsilon\backslash\varepsilon)^{\downarrow}$.

Internal Version of L

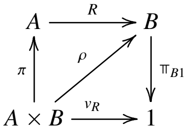
The whole lattice structure as well as the arrow operations can be recovered on the interval version of L.

$$\begin{aligned}
\text{zero} &= \Lambda(\perp\!\!\!\perp_{11}), \\
\text{one} &= \Lambda(\mathbb{I}_1), \\
\mathcal{M}_2 &= \Lambda(\pi;\varepsilon^{\smile} \sqcap \rho;\varepsilon^{\smile}), \\
\mathcal{J}_2 &= \Lambda(\pi;\varepsilon^{\smile} \sqcup \rho;\varepsilon^{\smile}), \\
\text{down} &= \Lambda(\varepsilon^{\smile\downarrow}), \\
\text{up} &= \Lambda(\varepsilon^{\smile\uparrow})
\end{aligned}$$

Covectorization

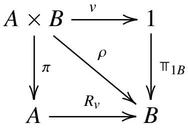
Given a relation $R : A \to B$ we may define a (co)vector $\mathrm{cov}(R) : A \times B \to 1$ by:

$$\mathrm{cov}(R) := (\pi; R \sqcap \rho); \top\!\!\!\!\top_{B1}$$



Given a (co)vector $v : A \times B \to 1$ we may define:

$$\mathrm{rel}(v) := \pi^{\smile}; (v; \top\!\!\!\!\top_{1B} \sqcap \rho)$$



We have

$$\mathrm{rel}(\mathrm{cov}(R)) = R \text{ and } \mathrm{cov}(\mathrm{rel}(v)) = v.$$

Covectorization II

We define

$$\begin{aligned} \mathrm{swap}_{AB} &:= \rho;\pi^{\smile} \sqcap \pi;\rho^{\smile}, \\ \mathrm{comp}_{ABC} &:= (\mathbb{I}_{(A\times B)\times(B\times C)} \sqcap \pi;\rho;\pi^{\smile};\rho^{\smile});(\pi;\pi;\pi^{\smile} \otimes \rho;\rho;\rho^{\smile}). \end{aligned}$$

Then we have:

Lemma

Let be $Q, R : A \to B$, and $S : B \to C$. Then we have:

1. $\mathrm{cov}(Q^{\downarrow}) = \mathrm{cov}(Q)^{\downarrow}$,
2. $\mathrm{cov}(Q^{\uparrow}) = \mathrm{cov}(Q)^{\uparrow}$,
3. $\mathrm{cov}(Q \sqcup R) = \mathrm{cov}(Q) \sqcup \mathrm{cov}(R)$,
4. $\mathrm{cov}(Q \sqcap R) = \mathrm{cov}(Q) \sqcap \mathrm{cov}(R)$,
5. $\mathrm{cov}(Q^{\smile}) = \mathrm{swap};\mathrm{cov}(Q)$,
6. $\mathrm{cov}(R;S) = \mathrm{comp}^{\smile};(\pi;\mathrm{cov}(R) \sqcap \rho;\mathrm{cov}(S))$.

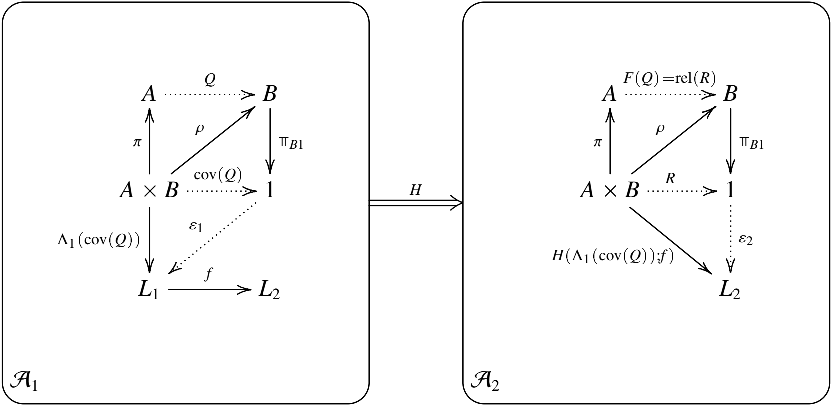
Change of Base I

We start with two arrow categories $\mathcal{A}_1$ and $\mathcal{A}_2$ each with a unit, relational products, and relational powers over the Heyting algebras L_1 and L_2, respectively, so that the Dedekind categories $\mathcal{A}_1^{\downarrow}$ and $\mathcal{A}_2^{\downarrow}$ are isomorphic with the same objects, i.e., we will require that there are functors $H : \mathcal{A}_1 \to \mathcal{A}_2$ and $H^{-1} : \mathcal{A}_2 \to \mathcal{A}_1$ that are the identity on objects, preserve all operations and constants of $\mathcal{A}_1^{\downarrow}$ resp. $\mathcal{A}_2^{\downarrow}$, and are inverse to each other.
Given a crisp map $f : L_1 \to L_2$ in $\mathcal{A}_1$, we define

$$F(Q) := \mathrm{rel}(H(\Lambda_1(\mathrm{cov}(Q));f);\varepsilon_2^{\smile})$$

for every $Q : A \to B$ in $\mathcal{A}_1$.

Change of Base II



Change of Base III

Theorem

F preserves converse, i.e., $F(Q^{\smile}) = F(Q)^{\smile}$ for all relations $Q : A \to B$.

Change of Base IV

$$
\begin{aligned}
\text{P1:} &\quad H((\mathbb{I}_{L_1} \sqcap \text{down}_1);f) = H(f);(\mathbb{I}_{L_2} \sqcap \text{down}_2),\\
\text{P2a:} &\quad H(\text{one}_1;f) = \text{one}_2,\\
\text{P2b:} &\quad \text{one}_2;H(f^{\smile}) = H(\text{one}_1).
\end{aligned}
$$

Theorem

If f satisfies **P1**, **P2a** *and* **P2b**, *then F extends H, i.e., $F(Q) = H(Q)$ for all crisp relations $Q : A \to B$.*

Change of Base V

P3: $H(\mathcal{J}_2^1;f) = H(f \otimes f);\mathcal{J}_2^2$.

Theorem

If f satisfies **P3**, *then F preserves binary joins, i.e., $F(Q \sqcup R) = F(Q) \sqcup F(R)$ for all relations $Q, R : A \rightarrow B$.*

Change of Base VI

P4: $H(\mathcal{M}_2^1;f) = H(f \otimes f);\mathcal{M}_2^2$.

Theorem

If f satisfies **P4**, *then F preserves binary meets, i.e., $F(Q \sqcap R) = F(Q) \sqcap F(R)$ for all relations $Q, R : A \to B$.*

Change of Base VII

P5: $H(\Lambda_1(\varepsilon_1^{\smile\downarrow};\varepsilon_1^{\smile});f) = \Lambda_2(H(\varepsilon_1^{\smile\downarrow};f));\mathcal{J}^2$.

Theorem

If f satisfies **P4** *and* **P5***, then F preserves composition, i.e., $F(R;S) = F(R);F(S)$ for all relations $R : A \to B$ and $S : B \to C$.*

Thank you for your attention.