L-fuzzy Concept Analysis using Arrow Categories

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Motivation

1. We would like to use L-fuzzy sets instead of regular sets in formal concept analysis leading to L-fuzzy formal concept analysis.
2. We would like to give a formulation of concept anaysis that covers L-fuzzy concept analysis as well as regular concept analysis.
3. We use the theory of arrow categories.

Prerequisites

We will se again the following concpets (see presentation on Wednesday):

1. Dedekind and arrow categories.
2. Unit.
3. Relational products and powers with the notation:

$$Q \otimes R = Q;\pi^{\smile} \sqcap R;\rho^{\smile}, \quad S \otimes T = \pi;S \sqcap \rho;T, \quad U \otimes V = \pi;U;\pi^{\smile} \sqcap \rho;V;\rho^{\smile}$$

Order theoretic concepts

Suppose $E : A \to A$ is an order relation.

1. $\mathrm{ubd}_E(R) := R^{\smile}\backslash E$, upper bounds of R with respect to E,
2. $\mathrm{lbd}_E(R) := R^{\smile}\backslash E^{\smile}$, lower bounds of R with respect to E,
3. $\mathrm{lub}_E(R) := \mathrm{ubd}_E(R) \sqcap \mathrm{lbd}_E(\mathrm{ubd}_E(R))$, least upper bound of R with respect to E,
4. $\mathrm{glb}_E(R) := \mathrm{lbd}_E(R) \sqcap \mathrm{ubd}_E(\mathrm{lbd}_E(R))$, greatest lower bound of R with respect to E.

An order is complete iff $\mathrm{lub}_E(R)$ is total for every R.

Splittings

Suppose $Q : A \to A$ is crisp partial equivalence relation, i.e., Q is crisp, symmetric ($Q^{\smile} = Q$), and idempotent ($Q; Q = Q$).

The splitting of Q is an object B together with a crisp relation $R : B \to A$ with $R; R^{\smile} = \mathbb{I}_C$ and $R^{\smile}; R = Q$.

Intuitively the object of the splitting representing the set of (existing) equivalence classes of Q.

Formal concept analysis

We consider a relation $I : G \to M$ between two objects G (Gegenstände / Objects) and M (Merkmale / Attributes).

Definition

Let $I : G \to M$ be a relation. Then $\triangle : L^G \to L^M$ and $\triangledown : L^M \to L^G$ are defined by

$$\triangle = \Lambda(\varepsilon \backslash I), \quad \text{and} \quad \triangledown = \Lambda(\varepsilon \backslash I^{\smile}).$$

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Lemma

Suppose G and M are sets, and $I : G \to M$ is an L-fuzzy relation. Then we have

$$\triangle(A, B) = 1 \iff \forall m \in M : B(m) = \left(\bigsqcap_{g \in G} A(g) \to I(g, m) \right),$$

$$\textit{and } \triangledown(B, A) = 1 \iff \forall g \in G : A(g) = \left(\bigsqcap_{m \in M} B(m) \to I(g, m) \right).$$

Formal concept analysis II

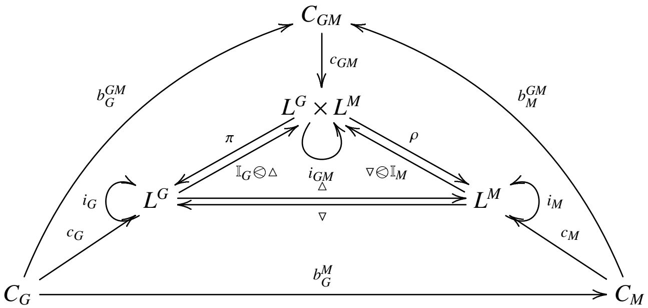
Definition

We define three relation $i_{GM} : L^G \times L^M \to L^G \times L^M$, $i_G : L^G \to L^G$, and $i_M : L^M \to L^M$ by

$$i_{GM} = \mathbb{I}_{L^G \times L^M} \sqcap (\vartriangle \otimes \triangledown); \mathrm{swap}, \quad i_G = \mathbb{I}_{L^G} \sqcap \vartriangle; \triangledown, \quad \text{and} \quad i_M = \mathbb{I}_{L^M} \sqcap \triangledown; \vartriangle.$$

Formal concept analysis III

Let $c_{GM} : C_{GM} \to L^G \times L^M$ (resp. $c_G : C_G \to L^G$ and $c_M : C_M \to L^M$) be the splitting of i_{GM} (resp. i_G and i_M).



Formal concept analysis IV

Definition

We define three relation $b_G^{GM} : C_G \to C_{GM}$, $b_M^{GM} : C_M \to C_{GM}$, and $b_G^{M} : C_G \to C_M$ by

1. $b_G^{GM} = c_G; (\mathbb{I}_{LG} \otimes \triangle); c_{GM}^{\smile}$,
2. $b_M^{GM} = c_M; (\nabla \otimes \mathbb{I}_{LM}); c_{GM}^{\smile}$,
3. $b_G^{M} = b_G^{GM}; (b_M^{GM})^{\smile}$.

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3. $b_G^M = b_G^{GM}; (b_M^{GM})^{\smile}$.

Theorem

The relations b_G^{GM}, b_M^{GM}, and b_G^M are bijective maps.

Formal concept analysis V

Definition

We define three relation $E_{GM} : C_{GM} \to C_{GM}$, $E_G : C_G \to C_G$, and $E_M : C_G \to C_M$ by

$$E_{GM} = c_{GM}; (\Omega \otimes \Omega^{\smile}); c_{GM}^{\smile}, \quad E_G = c_G; \Omega; c_G^{\smile}, \ \text{ and } \ E_M = c_M; \Omega^{\smile}; c_M^{\smile}.$$

Formal concept analysis V

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Theorem

The order relations E_{GM}, E_G and E_M are complete lattices, i.e., $\mathrm{lub}_{E_{GM}}(R)$ is total for every R.

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Theorem

The relations b_G^{GM}, b_M^{GM}, and b_G^M are complete lattice homomorphisms.

Attribute Implications

We define

$$A_I = (I/\varepsilon^{\smile})\backslash(I/\varepsilon^{\smile}).$$

Lemma

Suppose G and M are sets, and $I : G \to M$ is an L-fuzzy relation. Then we have

$$A_I(P,Q) = \bigsqcap_{g\in G}\left[\left(\bigsqcap_{m\in M} P(m) \to I(g,m)\right) \to \left(\bigsqcap_{m\in M} Q(m) \to I(g,m)\right)\right].$$

Attribute Implications II

Lemma

For the relation A_I we have

1. $A_I = \nabla; \Omega; \nabla^{\smile}$,
2. $A_I = \nabla; \triangle; \Omega^{\smile}$,
3. $A_I = (i_M; \Omega^{\smile}) \backslash \Omega^{\smile}$,
4. $A_I = (c_M; \Omega^{\smile}) \backslash (c_M; \Omega^{\smile})$,

Object classification

We assume that the attributes are of the form $M+1$ where M are the regular attributes and the unit 1 models a global decision attribute and define:

$$e = \Lambda(\iota;\varepsilon)^{\smile} \sqcap \pi_{L^M L};\Lambda(\kappa;\varepsilon)^{\smile} : L^M \to L^{M+1}.$$

Furthermore, we assume a positive context $I^+ : G \to M+1$ and the negative context $I^- : G \to M+1$.

Then we define:

$$d^+ = \Lambda(v);\Omega^{\smile};e;c^{+\smile};\pi_{L^{M+1}1} : 1 \to 1,$$
$$d^- = \Lambda(v);\Omega^{\smile};e;c^{-\smile};\pi_{L^{M+1}1} : 1 \to 1.$$

Future work

1. Show that the concept lattice is the smallest complete lattice generated by I.
2. Investigate the Duquenne-Guigues or canonical basis more in this context.
3. Use a t-norm like operation in the formulation.
4. ...

Thank you for your attention.